GRADUATE SCHOOL OF
FACULTY OF SCIENCE
KYOTO UNIVERSITY

Extreme Universe
A New Paradigm for Spacetime and Matter from Quantum Information
Grant-in-Aid for Transformative Research Areas (A)

Fock space localization in a perturbed Sachdev-Ye-Kitaev model

22 March 2022

The 17th Japan-Slovenia Seminar
on Nonlinear Science (online)

Masaki TEZUKA (Kyoto University)

Plan

- Sachdev-Ye-Kitaev model
  - Maximally chaotic quantum mechanical model

$$\hat{H} = \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

- SYK4+2
  - Departure from chaotic behavior

$$\hat{H} = \sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

- Quantitative analysis of Fock-space localization
  - Many-body transition point
  - Spectral statistics, inverse participation ratio
  - Ergodicity of the eigenstates

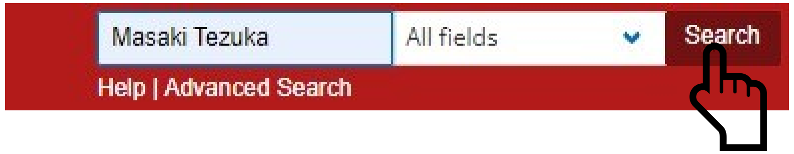
SYK model-related publications and collaborators

- Sachdev-Ye-Kitaev model
  - Proposal for experiment: PTEP 2017, 083I01 and arXiv:1709.07189
    - with Ippei Danshita and Masanori Hanada
  - Black Holes and Random Matrices: JHEP 1705(2017)118
    - with J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, and A. Streicher
  - Scrambling time: JHEP 1807(2018)124 with Hrant Gharibyan, M. Hanada, and S. H. Shenker
- SYK4+2
  - Chaotic-integrable transition: PRL **120**, 241603 (2018)
    - with Antonio M. García-García, Bruno Loureiro, and Aurelio Romero-Bermúdez
  - Characterization of quantum chaos: JHEP 1904(2019)082 and Phys. Rev. E **102**, 022213 (2020)
    - with Hrant Gharibyan, M. Hanada, and Brian Swingle
  - Related setups:
    - [short-range interactions] Phys. Rev. B **99**, 054202 (2019) with A. M. García-García
    - Phys. Lett. B **795**, 230 (2019) and J. Phys. A **54**, 095401 (2021) with Pak Hang Chris Lau, Chen-Te Ma, and Jeff Murugan

Also see our reply [PRL **126**, 109102 (2021)] to the comment by J. Kim and X. Cao [PRL **126**, 109101 (2021)]

- Quantitative analysis of Fock-space localization in SYK4+2
  - Many-body transition point and inverse participation ratio
    - Phys. Rev. Research **3**, 013023 (2021) with Felipe Monteiro, Tobias Micklitz, and Alexander Altland
  - Entanglement entropy
    - Phys. Rev. Lett. **127**, 030601 (2021) with F. Monteiro, A. Altland, David A. Huse, and T. Micklitz

Recent works **not** directly related to SYK


Masaki Tezuka

All fields

- [2112.12467](https://arxiv.org/abs/2112.12467) **Universal properties of dissipative Tomonaga-Luttinger liquids: A case study of a non-Hermitian XXZ spin chain**
  (with Kazuki Yamamoto, Masaya Nakagawa, Masahito Ueda, and Norio Kawakami)
- [2112.08763](https://arxiv.org/abs/2112.08763) **Topological Lifshitz Transitions, Orbital Currents, and Interactions in Low-dimensional Fermi Gases in Synthetic Gauge Fields**
  (with Chen-How Huang and Miguel A. Cazalilla) New J. Phys. (2022) in press
- [2111.03857](https://arxiv.org/abs/2111.03857) **Ferromagnetism in tilted fermionic Mott insulators**
  (with Kazuaki Takasan)
- [2110.03008](https://arxiv.org/abs/2110.03008) **Numerical evidence for a Haagerup conformal field theory**
  (with Tzu-Chen Huang, Ying-Hsuan Lin, Kantaro Ohmori, and Yuji Tachikawa)
- [2012.14609](https://arxiv.org/abs/2012.14609) **Local Operator Entanglement in Spin Chains**
  (with Eric Mascot, Masahiro Nozaki)
- Fuyuki Matsuda, Masaki Tezuka, and Norio Kawakami,
  Two-Dimensional Thouless Pumping of Ultracold Fermions in Obliquely Introduced Optical Superlattice, J. Phys. Soc. Jpn. **89**, 114708 (2020)

Sachdev-Ye-Kitaev model

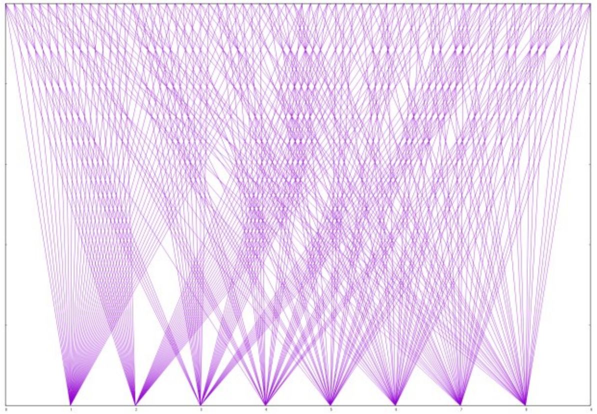
[Kitaev, talks at KITP (2015)]
[Sachdev, PRX (2015)]
[Sachdev and Ye, PRL (1993)]

$$\hat{H} = \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

$\hat{\chi}_{a=1,2,\ldots,N}$: N Majorana fermions ($\{\hat{\chi}_a, \hat{\chi}_b\} = 2\delta_{ab}$)

J_{abcd} : independent Gaussian random couplings ($\overline{J_{abcd}{}^2} = J^2$, $\overline{J_{abcd}} = 0$)

$\binom{N}{4} \cong \frac{N^4}{24}$ terms



N Majorana fermions

- Solvable in the large-N limit

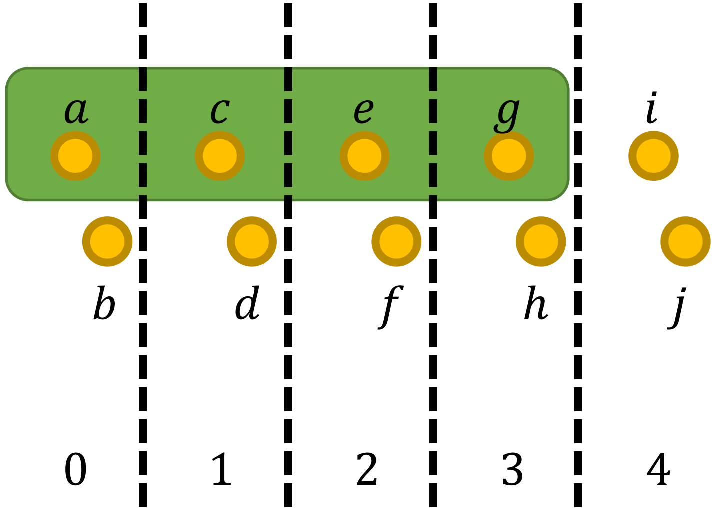
[Maldacena, Shenker, and Stanford, JHEP08(2016)106]

- Maximally chaotic ($\lambda_{\mathrm{Lyapunov}} \overset{\text{low } T}{\to} 2\pi k_{\mathrm{B}}T/\hbar$: chaos bound)
- Correspondence to 1+1d gravity, random matrix

Reviews e.g. [D. Chowdhury, A. Georges, O. Parcollet, and S. Sachdev, 2109.05037];
[M. Tezuka, Kotai Butsuri (in Japanese) March 2022]

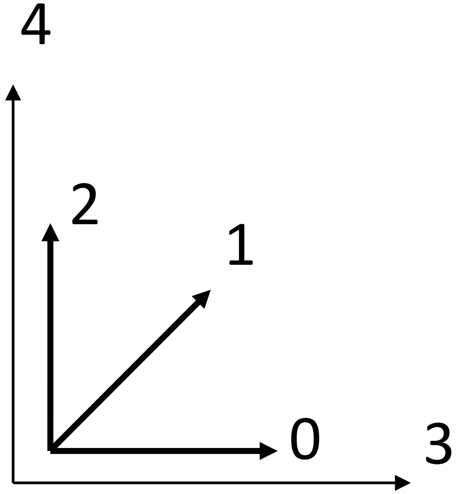
One term of the 10-Majorana fermion SYK$_{q=4}$

$\chi_a \chi_c \chi_e \chi_g$

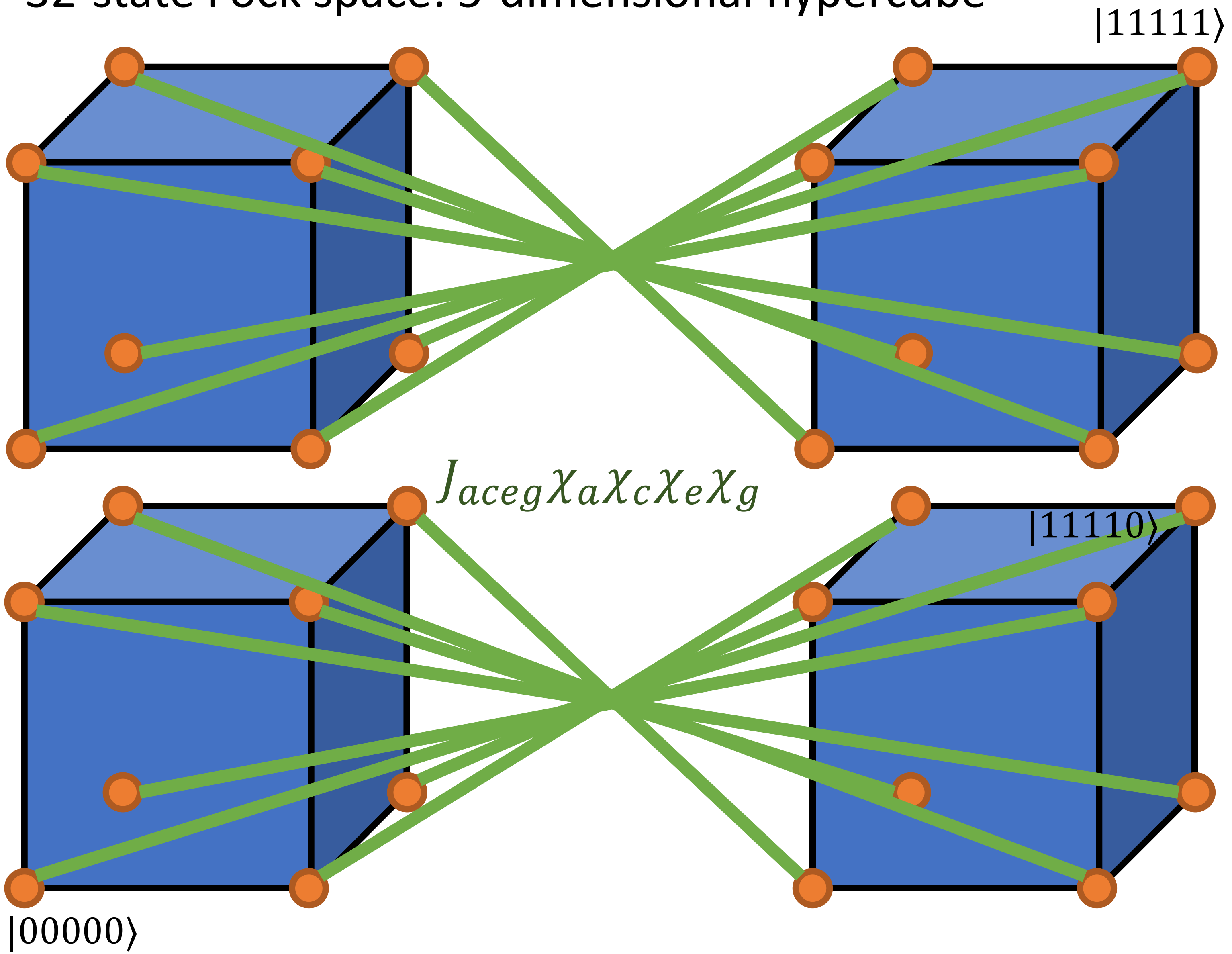


5 qubits

$\binom{10}{4}$ = 210 terms



32-state Fock space: 5-dimensional hypercube

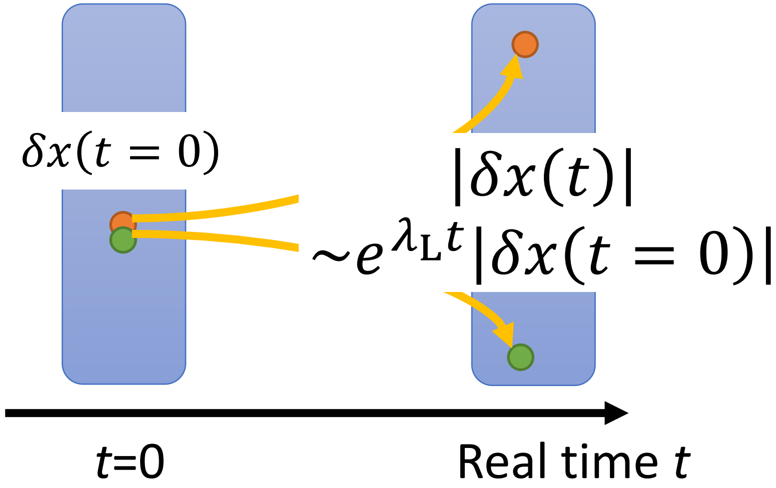


Lyapunov exponent and out-of-time-order correlators (OTOC)

$$F(t) = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle \quad W(t) = \mathrm{e}^{iHt} W \mathrm{e}^{-iHt}$$

Classical chaos:

Infinitesimally different initial coords



λ_{L}: Lyapunov exponent

$$\left(\frac{\partial x(t)}{\partial x(0)}\right)^2 = \{x(t), p(0)\}_{\mathrm{PB}}{}^2 \rightarrow e^{2\lambda_{\mathrm{L}} t}$$

Quantum dynamics:

$$C_T(t) = \langle [\hat{x}(t), \hat{p}(0)]^2 \rangle$$

For operators V and W, consider

$$C(t) = \langle |[W(t), V(t=0)]|^2 \rangle = \langle W^\dagger(t) V^\dagger(0) W(t) V(0) \rangle + \cdots$$

[Wiener 1938][Larkin & Ovchinnikov 1969]

OTOC $\sim e^{2\lambda_{\mathrm{L}} t}$ at long times, $\lambda_{\mathrm{L}} > 0$: chaotic

"Black holes are fastest quantum scramblers"

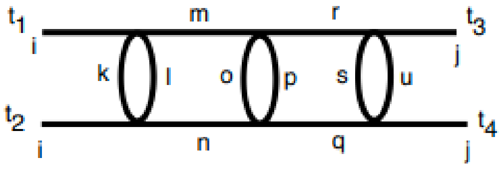
[P. Hayden and J. Preskill 2007] [Y. Sekino and L. Susskind 2008]
[Shenker and Stanford 2014]

$\lambda_{\mathrm{L}} \leq 2\pi k_{\mathrm{B}} T/\hbar$ (chaos bound)

[J. Maldacena, S. H. Shenker, and D. Stanford, JHEP08(2016)106]

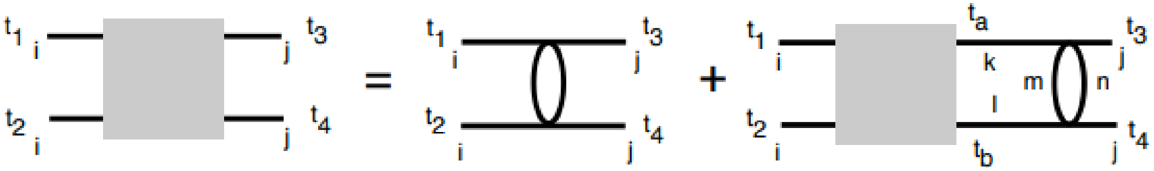
Out-of-time-ordered correlators (OTOCs)

$$\langle \hat{\chi}_i(t_1)\hat{\chi}_i(t_2)\hat{\chi}_j(t_3)\hat{\chi}_j(t_4)\rangle$$

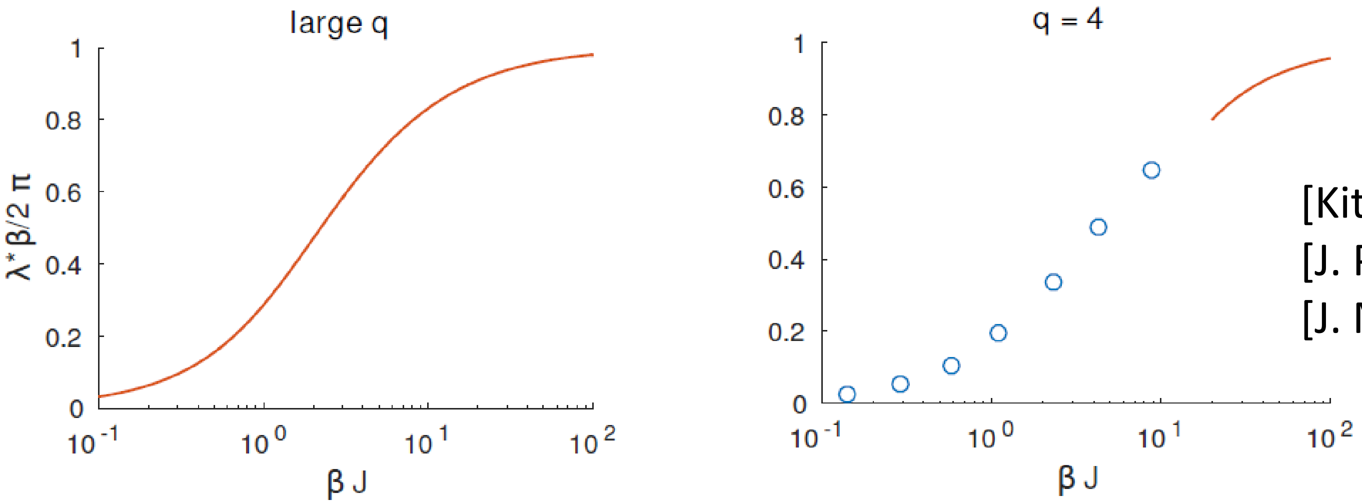


(a)

Regularized OTOC can be calculated for large-*N* SYK model, satisfies the chaos bound $\lambda_{\mathrm{L}} = 2\pi k_{\mathrm{B}} T/\hbar$ at low *T* limit



$$\Gamma(t_1,t_2,t_3,t_4) = \Gamma_0(t_1,t_2,t_3,t_4) + \int dt_a dt_b\, \Gamma(t_1,t_2,t_a,t_b)K(t_a,t_b,t_3,t_4)$$

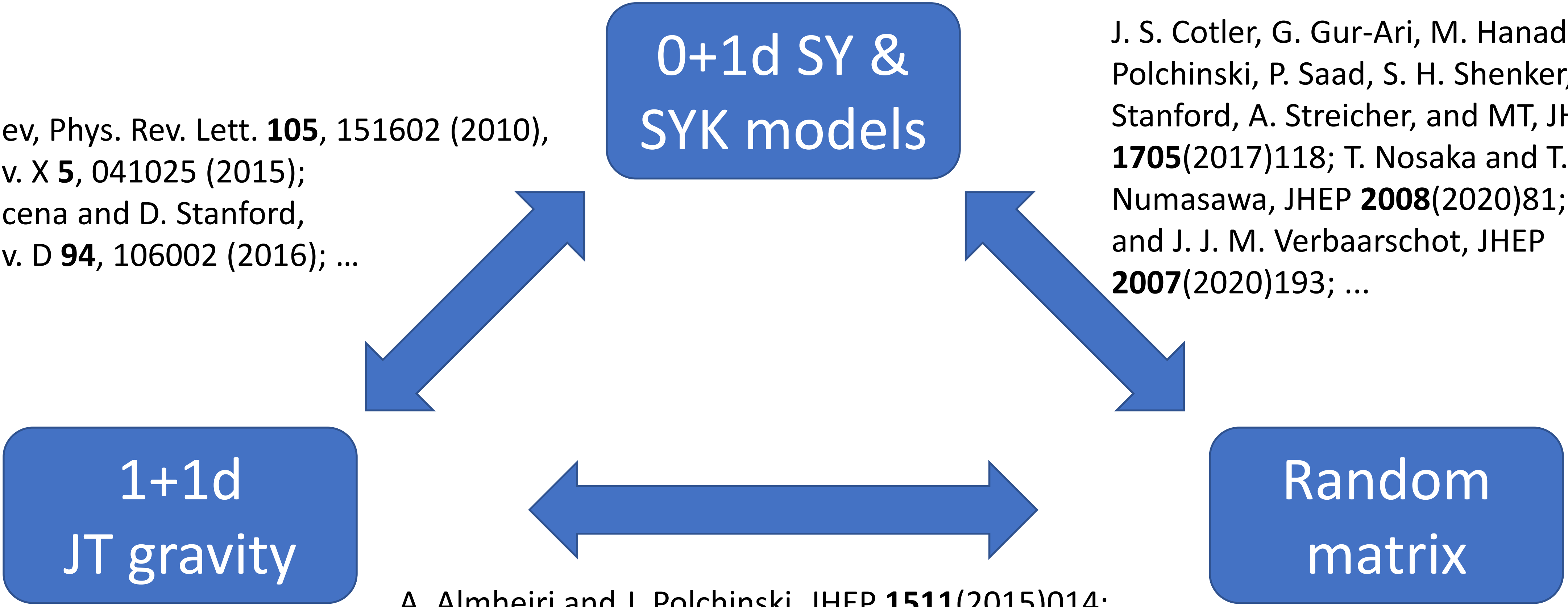


[Kitaev's talks]

[J. Polchinski and V. Rosenhaus, JHEP 1604 (2016) 001]

[J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)]

Maximally chaotic systems



0+1d SY & SYK models

1+1d JT gravity

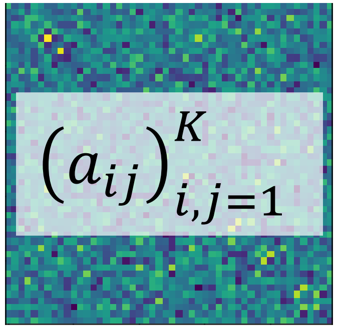
Random matrix

S. Sachdev, Phys. Rev. Lett. **105**, 151602 (2010), Phys. Rev. X **5**, 041025 (2015); J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016); …

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and MT, JHEP **1705**(2017)118; T. Nosaka and T. Numasawa, JHEP **2008**(2020)81; Y. Jia and J. J. M. Verbaarschot, JHEP **2007**(2020)193; ...

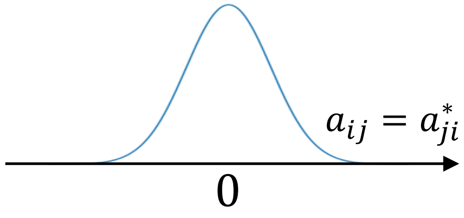
A. Almheiri and J. Polchinski, JHEP **1511**(2015)014;
P. Saad, S. H. Shenker, and D. Stanford, arXiv:1903.11115;
D. Stanford and E. Witten, arXiv:1907.03363; …

Gaussian random matrices



$$\left(a_{ij}\right)_{i,j=1}^{K}$$

Gaussian distribution



β: Dyson index

Real ($\beta = 1$): Gaussian Orthogonal Ensemble (GOE)
Complex ($\beta = 2$): G. Unitary E. (GUE)
Quaternion ($\beta = 4$): G. Symplectic E. (GSE)

$$\text{Density} \propto e^{-\frac{\beta K}{4}\mathrm{Tr}H^2} = \exp\left(-\frac{\beta K}{4}\sum_{i,j}^{K}\left|a_{ij}\right|^2\right)$$

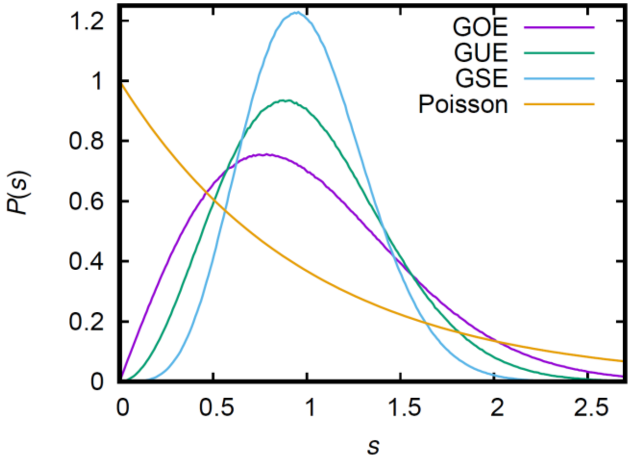
Joint distribution function for eigenvalues $\{e_j\}$

Level repulsion

$$p(e_1, e_2, \ldots, e_K) \propto \prod_{1\le i<j\le K}\left|e_i - e_j\right|^{\beta}\prod_{i=1}^{K} e^{-\beta K e_i^2/4}$$

Distribution of normalized level separation $s_j = \frac{e_{j+1} - e_j}{\Delta(\bar{e})}$

GOE/GUE/GSE: $P(s) \propto s^{\beta}$ at small s, has e^{-s^2} tail



Uncorrelated: $P(s) = e^{-s}$ (Poisson distribution)

Neighboring gap ratio

$$r = \frac{\min(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}$$

	Uncorrelated	GOE	GUE	GSE
$\langle r\rangle$	2log 2 – 1 = 0.38629…	0.5307(1)	0.5996(1)	0.6744(1)

[Y. Y. Atas *et al.* PRL 2013]

➔ SYK model: level correlation ($P(s), P(r), \langle r\rangle$, etc.) indistinguishable from corresponding Gaussian ensemble

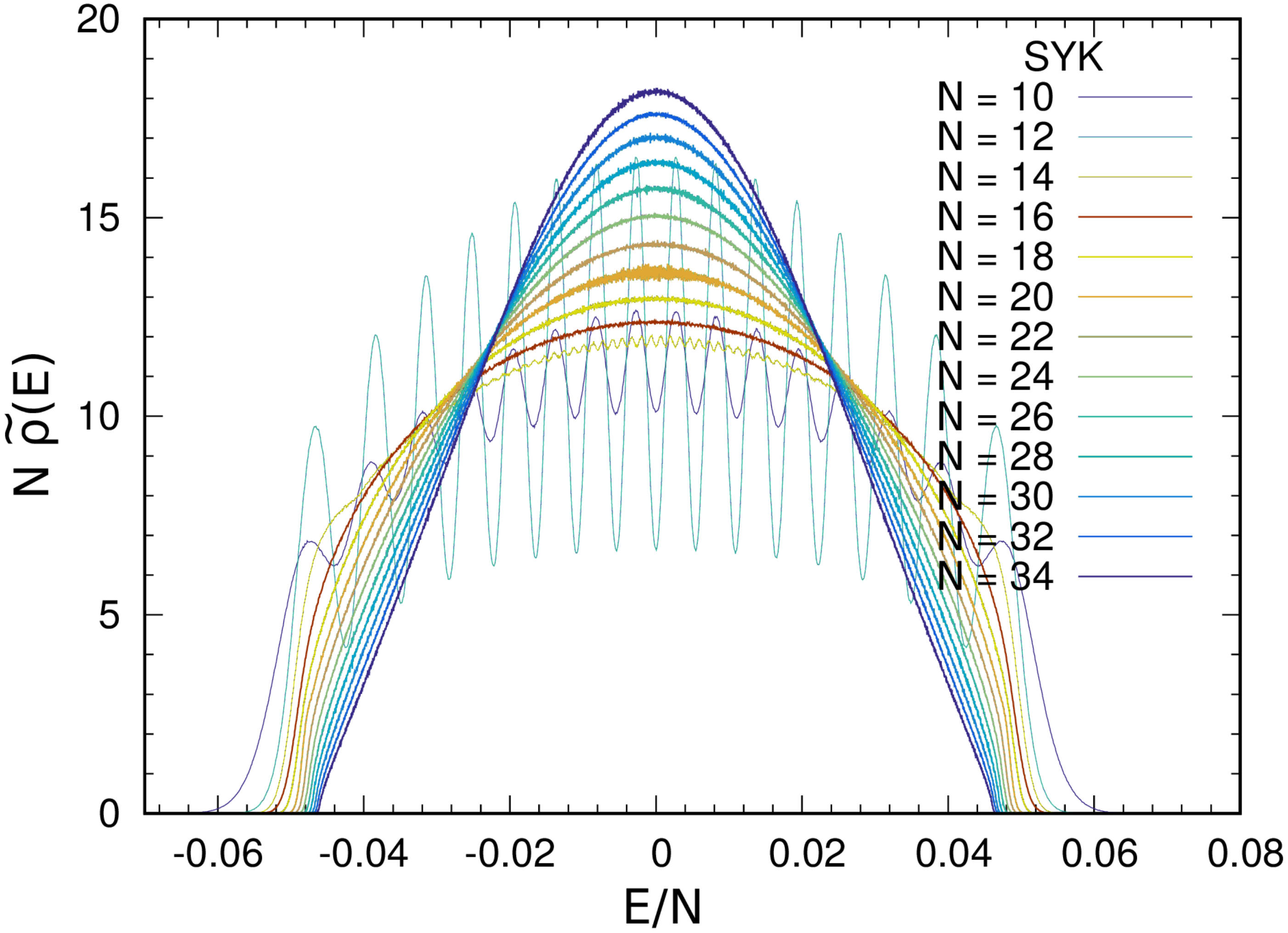
Majorana SYK4 with
$N \equiv 0 \pmod 8$: GOE
$N \equiv 2, 6 \pmod 8$: GUE
$N \equiv 4 \pmod 8$: GSE

[Fidkowski and Kitaev PRB 2010, 2011]
[You, Ludwig, and Xu PRB 2017]

J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and MT, JHEP **1705**(2017)118

Eigenvalue spectrum

$$\hat{H} = \sum_{1 \le a < b < c < d \le N_{\mathrm{M}}} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d,$$



cf. DoS for large *N* [A. M. García-García and J. J. M. Verbaarschot: PRD **94**, 126010 (2016)]

Plan

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$$\hat{H} = \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

- SYK4+2
  - Departure from chaotic behavior

$$\hat{H} = \sum_{1 \le a < b < c < d}^{N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d + i \sum_{1 \le a < b}^{N} K_{ab} \hat{\chi}_a \hat{\chi}_b$$

- Quantitative analysis of Fock-space localization
  - Many-body transition point
  - Spectral statistics, inverse participation ratio
  - Ergodicity of the eigenstates

A. M. García-García, A. Romero-Bermúdez, B. Loureiro, and MT, Phys. Rev. Lett. **120**, 241603 (2018)

SYK$_{4+2}$

Also see our reply [PRL **126**, 109102 (2021)] to the comment by J. Kim and X. Cao [PRL **126**, 109101 (2021)]

Q.: Minimum requirements for chaotic behavior? (→ gravity interpretation?)
Study a simple model with analytical + numerical methods

$$\hat{H} = \overbrace{\sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}^{\text{SYK}_4} + \overbrace{i\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}^{\text{SYK}_2}$$

Gaussian random couplings

J_{abcd}: average 0, standard deviation $\frac{\sqrt{6}J}{N^{3/2}}$

K_{ab}: average 0, standard deviation $\frac{K}{\sqrt{N}}$

$J = 1$: unit of energy

SYK$_4$ as unperturbed Hamiltonian,
K controls the strength of SYK$_2$ (one-body random term, solvable)

Here we take (GUE)
$N \equiv 2,6 \pmod 8$

Both terms respect charge parity in complex fermion description

➔ Full numerical exact diagonalization (ED) of $2^{N/2-1}$-dimensional matrix, $N \lesssim 34$ possible

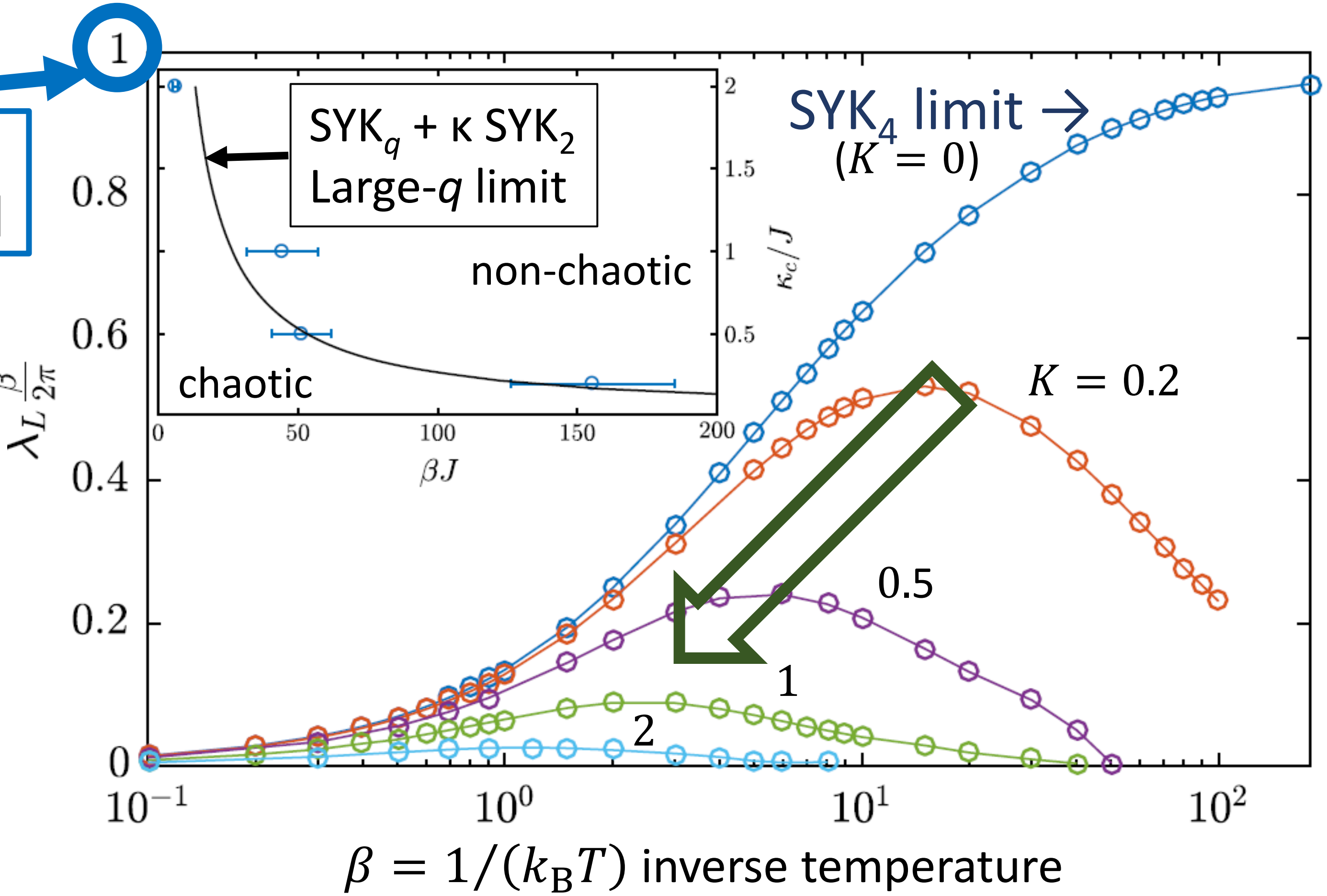
Large-*N* calculation for Out-of-Time Order Correlator (OTOC)

$$\hat{H} = \underbrace{\sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}_{\text{SYK}_4} + i \underbrace{\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}_{\text{SYK}_2}$$

K_{ab}: standard deviation $\frac{K}{\sqrt{N}}$

Chaos bound [Maldacena, Shenker, and Stanford 2016]

normalized Lyapunov exponent

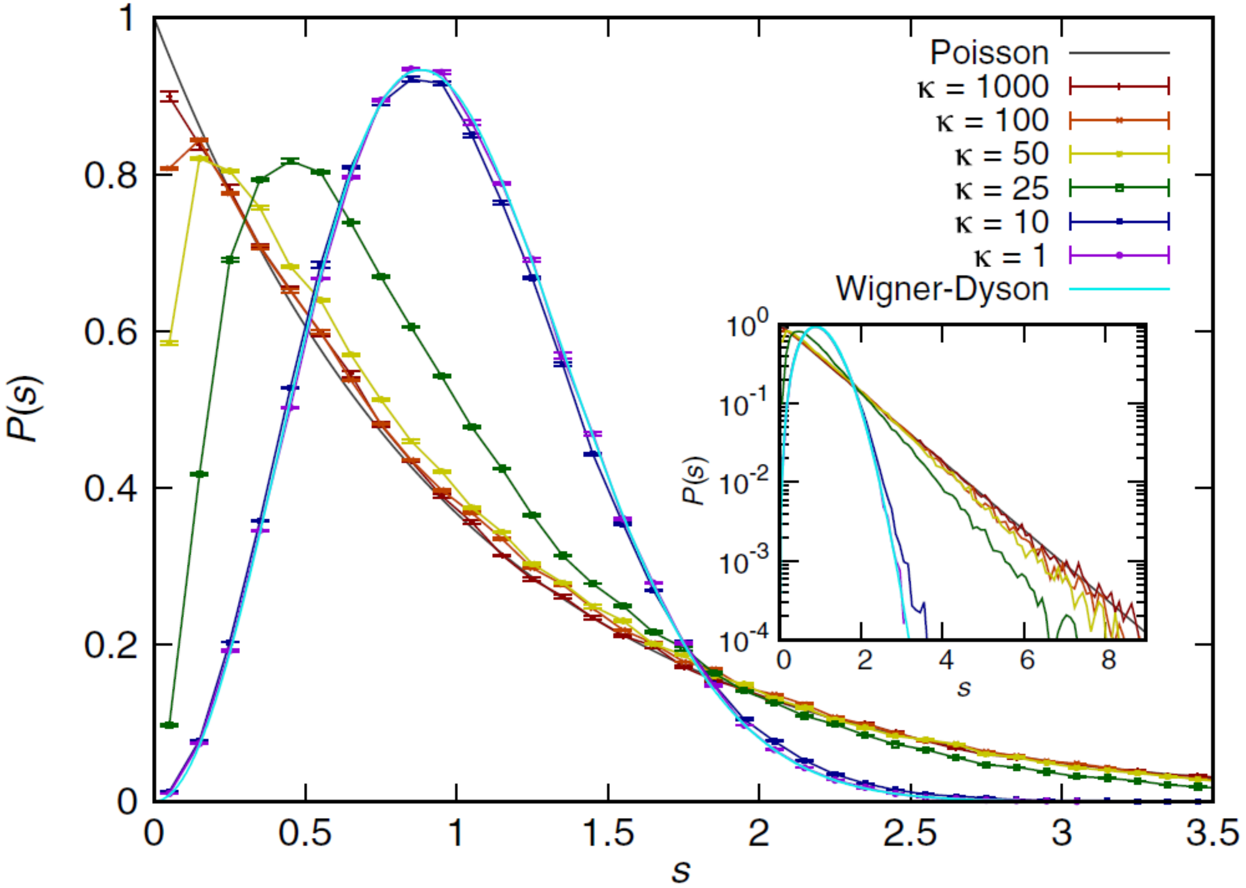


Antonio M. Garcia-Garcia, Bruno Loureiro, Aurelio Romero-Bermudez, and MT, PRL **120**, 241603 (2018)

Deviation from the chaos bound as SYK$_2$ component is introduced

PRL **120**, 241603 (2018)

RMT-like behavior lost as SYK$_2$ term is introduced



N=30, Central 10 % of eigenvalues

$P(s)$: level spacing distribution

Ratio of consecutive level spacing $E_{i+1} - E_i$
to the local mean level spacing Δ
(requires unfolding of the spectrum)

SYK$_4$ limit (small *K*):
Obeys random matrix theory (RMT)
(GUE (Gaussian Unitary Ensemble) if $N \equiv 2{,}6 \pmod 8$)

SYK$_2$ (large *K*): Poisson (e^{-S})

Also see: T. Nosaka, D. Rosa, and J. Yoon, JHEP **1809**, 041 (2018) for other symmetry cases
cf. A. V. Lunkin, K. S. Tikhonov, and M. V. Feigel’man, PRL 121, 236601 (2018); Y. Yu-Xiang, F. Sun, J. Ye, and W. M. Liu, 1809.07577, …

PRL **120**, 241603 (2018)

$\mathrm{SYK}_{q\geq 4} + \mathrm{SYK}_2$: breakdown of chaos

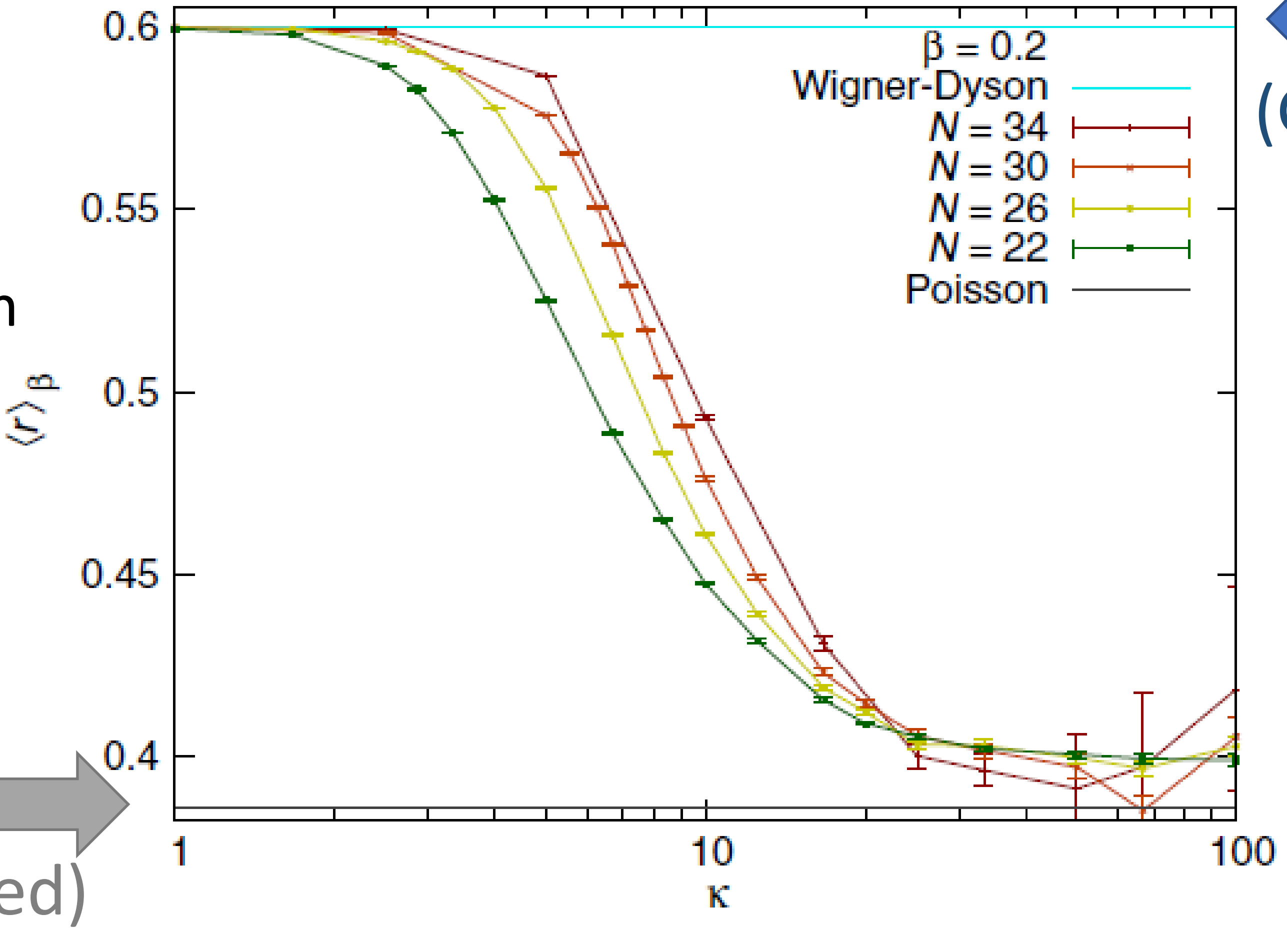
$$\hat{H} = \overset{\text{SYK}_4}{\sum_{1\leq a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d} + i \overset{\text{SYK}_2}{\sum_{1\leq a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}$$

K_{ab}: standard deviation = $\kappa/\sqrt{N}$

GUE

(Gaussian Unitary Ensemble)

Averaged ratio between neighboring energy level separations



Poisson

(uncorrelated)

We consider N Majorana fermions with normalization $\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$ here

Deviation from Gaussian random matrix as SYK_2 component is introduced

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Many-body localization

ETH: “(almost) all eigenstates are thermal (expectation values of operators = microcanonical average)”

- Anderson localization: concept in non-interacting systems
  - Localization of wavefunctions due to scatterings at impurities
  - Many experiments in cold atom gases, optical fibers, etc.
- MBL: does localization occur in interacting systems?

  [Gornyi, Mirlin, Polyakov 2005, Basko, Aleiner, Altshuler 2006, Oganesyan and Huse 2007, … many others]
  - Memory of initial conditions remains accessible at long times
  - Reduced density matrix on a subsystem does not approach a thermal one
  - Energy eigenstates do not obey Eigenstate Thermalization Hypothesis (ETH)
  - Area law, rather than volume law, of entanglement entropy
  - “Standard model”: spin-1/2 Heisenberg model + random field in *z* direction
    - Much debate on the location of the localization transition
    - See e.g. Avalanches [Crowley and Chandran PRR 2020] [A. Morningstar et al., 2107.05642] No localization? [Sels and Polkovnikov PRE 2021]

$$\hat{H} = \sum_i^N \hat{S_i} \cdot \widehat{S_{i+1}} + \sum_i^N h_i \widehat{S_i^z}$$

$h_i \in [-h, h]$ uniform distribution

F. Monteiro, T. Micklitz, MT, and A. Altland, Phys. Rev. Research **3**, 013023 (2021)

Our model and choice of basis

$$\mathrm{SYK}_4 + \delta\ \mathrm{SYK}_2$$

$$\hat{H} = -\sum_{1\le a<b<c<d}^{N=2N_{\mathrm{D}}} J'_{abcd}\hat{\psi}_a\hat{\psi}_b\hat{\psi}_c\hat{\psi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\psi}_a\hat{\psi}_b$$

Block-diagonalize the SYK_2 part
(the skew-symmetric matrix (K_{ab}) has eigenvalues $\pm v_j$)

$$\hat{H} = -\sum_{1\le a<b<c<d}^{2N_{\mathrm{D}}} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{j=1}^{N_{\mathrm{D}}} v_j\hat{\chi}_{2j-1}\hat{\chi}_{2j}$$

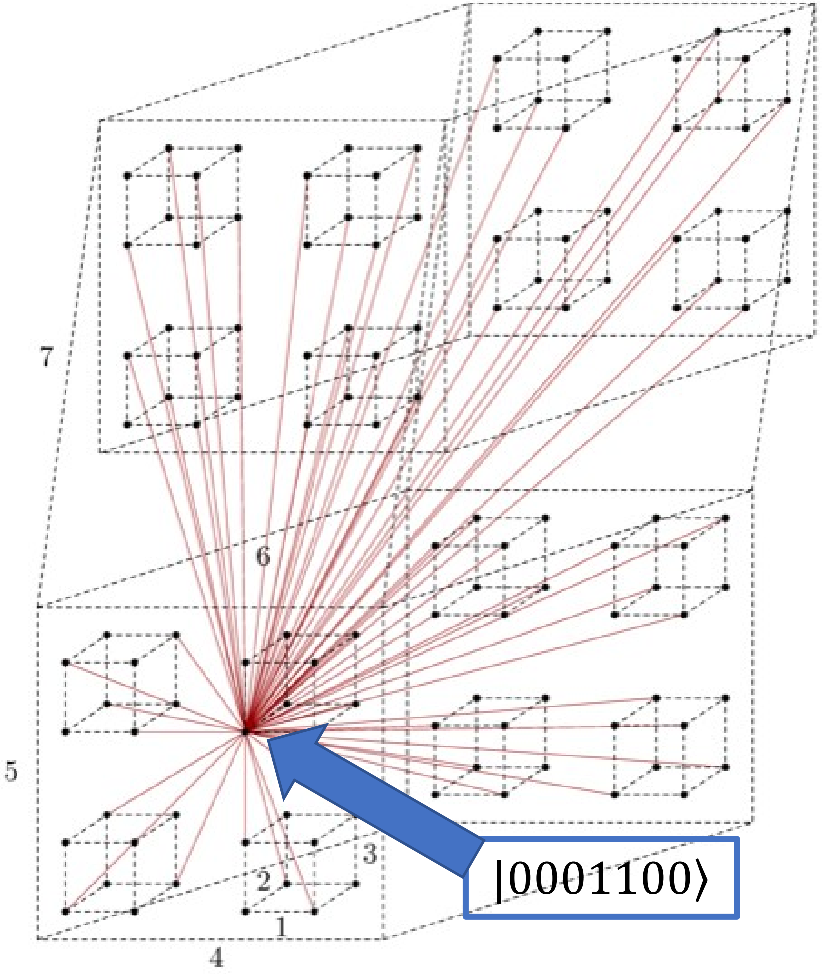
Normalization of J_{abcd}, v_j :
SYK_4 bandwidth = 1,
Width of v_j distribution = δ

We choose $\{\hat{\psi}_a, \hat{\psi}_b\} = \{\hat{\chi}_a, \hat{\chi}_b\} = 2\delta_{ab}$ as the normalization for the $N = 2N_{\mathrm{D}}$ Majorana fermions. For $\hat{c}_j = \frac{1}{2}\left(\hat{\chi}_{2j-1} + \mathrm{i}\hat{\chi}_{2\mathrm{j}}\right)$ we have $\{\hat{c}_i, \hat{c}_j^{\dagger}\} = \delta_{ij}$.

F. Monteiro, T. Micklitz, MT, and A. Altland, Phys. Rev. Research **3**, 013023 (2021)

Our model and choice of basis

$N = 2N_{\mathrm{D}} = 14$: $2^7 = 128$ states



$|0001100\rangle$

Basis diagonalizing the complex fermion number operators $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j$ → Sites: the $2^{N_{\mathrm{D}}}$ vertices of an N_{D}-dim. hypercube.

$$\hat{c}_j = \frac{1}{2}\left(\hat{\chi}_{2j-1} + \mathrm{i}\hat{\chi}_{2\mathrm{j}}\right)$$

$$\widehat{H} = -\sum_{1\le a<b<c<d}^{2N_{\mathrm{D}}} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le j\le N}^{N_{\mathrm{D}}} v_j\hat{\chi}_{2j-1}\hat{\chi}_{2j}$$

$$= -\sum_{1\le a<b<c<d}^{2N_{\mathrm{D}}} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + \sum_{1\le j\le N}^{N_{\mathrm{D}}} v_j\left(2\hat{n}_j - 1\right)$$

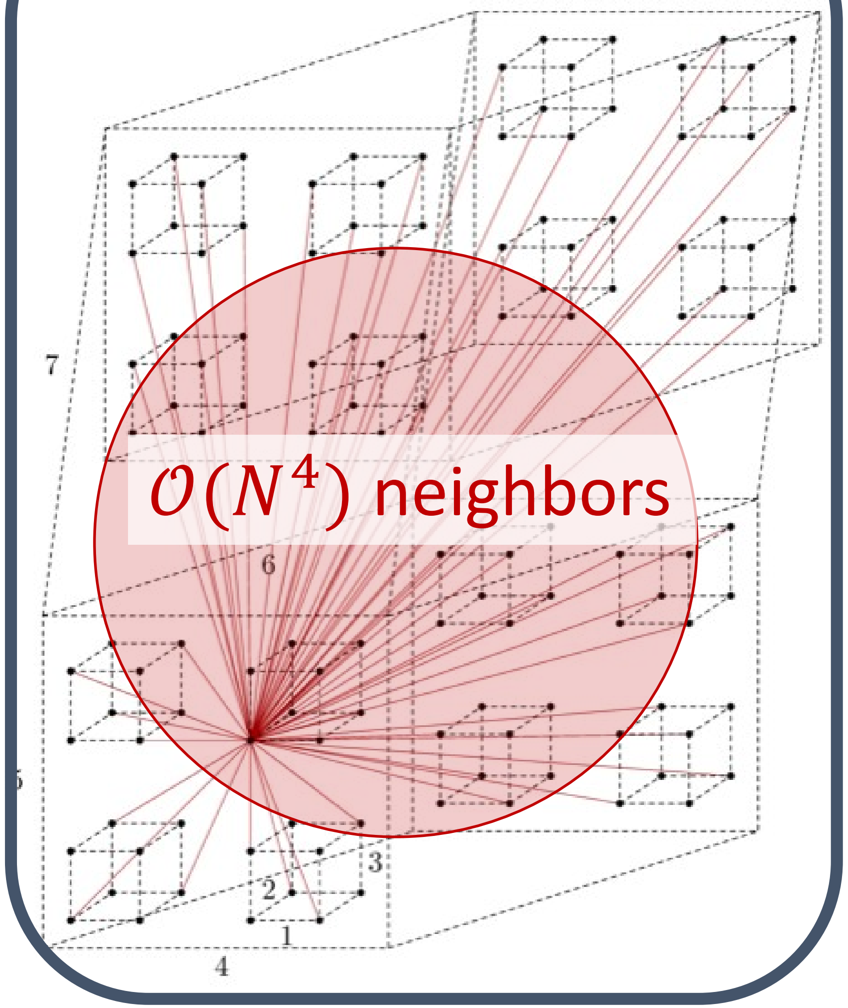
Each term of SYK_4 connects vertices with distance = 0, 2, 4.

For $N = 14$, each vertex is directly connected with 1 (distance=0, itself) + 21 (distance=2) + 35 (distance=4) vertices out of the possible $2^N = 128$ (64 per parity).

F. Monteiro, T. Micklitz, MT, and A. Altland, Phys. Rev. Research **3**, 013023 (2021)

Our model and choice of basis

2^{N_D} Fock states



Basis diagonalizing the complex fermion number operators $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j$ → Sites: the 2^{N_D} vertices of an N_D-dim. hypercube.

$\mathrm{SYK}_4 + \delta\ \mathrm{SYK}_2$ SYK$_4$ bandwidth = 1 width of v_j dist. = δ

hopping site energy

$$\hat{H} = -\boxed{\sum_{1\le a<b<c<d}^{2N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d} + \boxed{\sum_{1\le j\le N}^{N} v_j(2\hat{n}_j - 1)}$$

Each term of SYK_4 connects vertices with distance = 0, 2, 4.

For $N = 34$, each vertex is directly connected with
1 (distance=0, itself) + 136 (distance=2) + 2380 (distance=4) vertices out of the possible $2^{N/2} = 131072$ (65536 per parity).

See T. Micklitz, F. Monteiro, and A. Altland, Phys. Rev. Lett. **123**, 125701 (2019).

Picture for Fock space localization

$$\hat{H}_0 = \frac{1}{4!}\sum_{i,j,k,l=1}^{2N} J_{ijkl}\hat{\chi}_i\hat{\chi}_j\hat{\chi}_k\hat{\chi}_l,$$

$$\hat{H}_V = \gamma\sum_{n}^{D} v_n |n\rangle\langle n|,$$

$\left\langle \left|J_{ijkl}\right|^2 \right\rangle = \frac{6J^2}{(2N)^3}$, hopping amplitude $t = \left|J_{ijkl}\right| \sim JN^{-\frac{3}{2}}$

$\left| n = \sum_{1\leq j\leq N} 2^{j-1} n_j \right\rangle$

v_n: operator diagonal in the occupation number basis

Here (on this slide only): drawn from a box distribution of width $\Delta \sim N^{\alpha}$

SYK$_4$ bandwidth $\sim JN^{\frac{1}{2}}$ set to unity, $J \sim N^{-\frac{1}{2}}$

Characteristic hopping amplitude by SYK$_4$ Hamiltonian: $t \sim JN^{-\frac{3}{2}} \sim N^{-2}$

Hybridization of two nearest neighbors occur for $\frac{\kappa}{\Delta}$ of $\mathcal{O}(N^4)$ neighbors

Number of neighbors in resonance

Typical level broadening: κ

By Fermi's golden rule, $\kappa \sim |t|^2 \frac{N^4\kappa}{\Delta}/\kappa$.

Broadened energy

$\therefore \kappa \sim \frac{1}{\Delta} \sim N^{-\alpha}$ by self-consistency;

$\frac{\kappa}{\Delta} \sim \Delta^{-2} \sim N^{-2\alpha}$.

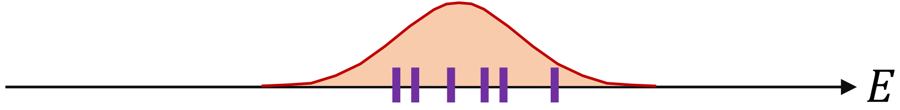
Typical # of resonant neighboring levels: $\frac{N^4}{\Delta^2} \sim N^{4-2\alpha}$. If $\alpha > 2$, localization (fragmentation of the Fock space) is possible

What happens in the case that v_n are given by SYK$_2$ eigenvalues?

Four regimes of disorder strengths

I

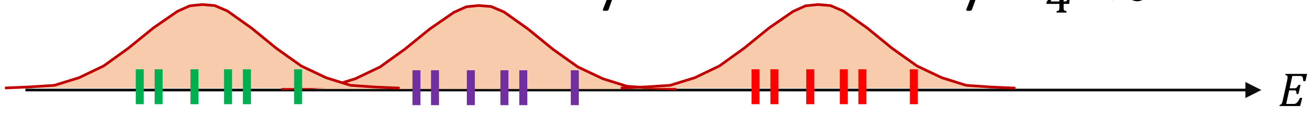
- Typical energy difference between arbitrary pair of sites $\lesssim 1$



$N_{\mathrm{D}}^{-1/2}$

II

- Typical energy difference > 1, but difference between sites directly connected by $\hat{H}_4 \lesssim 1$



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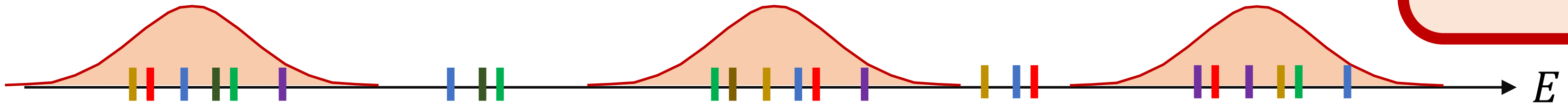
III

- Difference between sites connected by $\hat{H}_4 > 1$



δ_{c}

IV

- Fock space localization ($\delta_{\mathrm{c}} \sim N_{\mathrm{D}}^2 \ln N_{\mathrm{D}}$ for Bethe lattice)

[Altshuler, Gefen, Kamenev, and Levitov, PRL **78**, 2803 (1997)]
logarithmic correction from hybridization beyond nearest neighbors

δ

$$\hat{H} = \hat{H}_4 + \hat{H}_2 \qquad \hat{H}_2 = \sum_{1\le j\le N}^{N} v_j(2\hat{n}_j - 1)$$

width of v_j dist. = δ

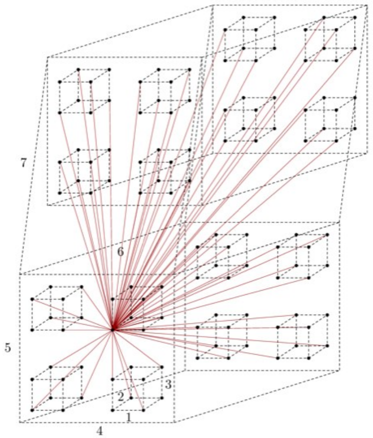
Site energy of site #m:

$$\epsilon_{(m=\sum_{1\le j\le N} 2^{j-1} n_j)} = \sum_{1\le j\le N}^{N} (-1)^{n_j-1} v_j$$

Width of ϵ_m dist. $= \sqrt{N_{\mathrm{D}}}\,\delta$

$$\hat{H}_4 = -\sum_{1\le a<b<c<d}^{2N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

SYK$_4$ bandwidth = 1



Diagnostic quantities: Moments of wave functions and spectral two-point correlation function

- Moments of eigenstate wave functions

$$I_q = \nu^{-1}\sum_{n,\psi}\left\langle |\langle\psi|n\rangle|^{2q}\,\delta\left(E_\psi\right)\right\rangle_J$$

with average density of states at band center

$$\nu = \nu(E \simeq 0), \nu(E) = \sum_{\psi}\left\langle \delta\left(E - E_\psi\right)\right\rangle_J$$

➔Parametrizes localization, allows comparison with numerics

$I_2 = \nu^{-1}\sum_{n,\psi}\left\langle |\langle\psi|n\rangle|^{4}\,\delta\left(E_\psi\right)\right\rangle_J$:

inverse participation ratio (IPR), $\frac{1}{D} \leq I_2 \leq 1$

Equal weights

Single non-zero element

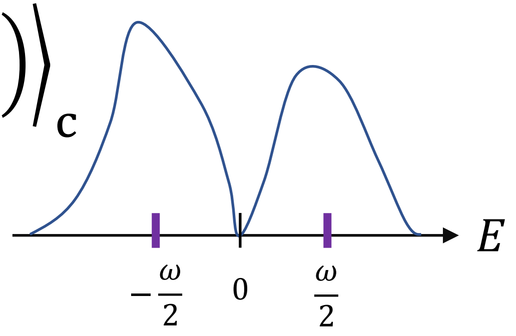
D: dimension of $\{|n\rangle\} = 2^{N-1}$

- Spectral two-point correlation function

$$K(\omega) = \nu^{-2}\left\langle \nu\left(\frac{\omega}{2}\right)\nu\left(-\frac{\omega}{2}\right)\right\rangle_{\mathrm{c}}$$

c: connected part

$\langle AB\rangle_c = \langle AB\rangle_J - \langle A\rangle_J \langle B\rangle_J$



➔ Reflects level repulsion if the spectrum is random matrix-like

We calculate these quantities for large *N* and compare against numerical results

Analytical results

Method: Exact matrix integral representation;
mapping to a supersymmetric sigma model;
saddle point equations; effective medium approximation

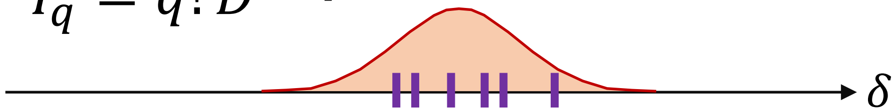
PRR 3, 013023 (2021)

$(N_{\mathrm{D}} = \frac{N}{2}, c = O(1), D = 2^{N_{\mathrm{D}}-1})$

I

- I: Average density of states (ADoS) at band center $\nu = cD$
- $I_q = q!\,D^{1-q}$

Fully delocalized

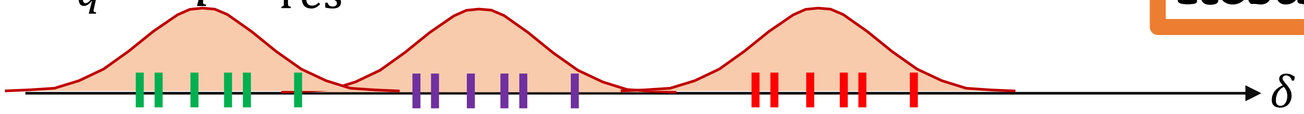


$N_{\mathrm{D}}^{-1/2}$

II

- II: ADoS $\nu = \frac{cD}{\sqrt{N_{\mathrm{D}}}\delta}$, spread of wave functions $D_{\mathrm{res}} \simeq \frac{D}{\sqrt{N_{\mathrm{D}}}\delta}$
- $I_q = q!\,D_{\mathrm{res}}^{1-q}$

Restricted



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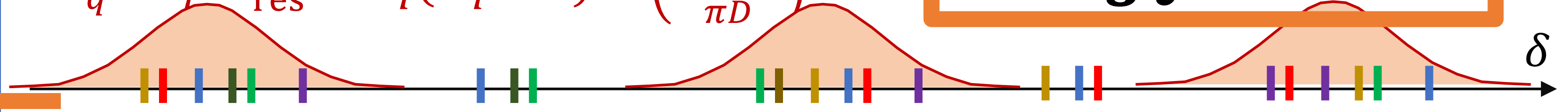
III

- III: ADoS $\nu = \frac{cD}{\sqrt{N_{\mathrm{D}}}\delta}$, spread of wave functions $D_{\mathrm{res}} \simeq \frac{D}{\sqrt{N_{\mathrm{D}}}\delta^2}$
- $I_q = q!\,D_{\mathrm{res}}^{1-q} = q(2q-3)!!\left(\frac{4\sqrt{N}\delta^2}{\pi D}\right)^{q-1}$

Strongly restricted



δ_{c}

IV

$$\delta_{\mathrm{c}} = \frac{N_{\mathrm{D}}^2}{4\sqrt{3}}\log N_{\mathrm{D}} \text{ for large } N$$

- IV: All eigenstates localized to $\mathcal{O}(1)$ sites

δ

Eigenenergy spectral statistics (for odd N case for simplicity)

$$\widetilde{K}(s) = 1 - \frac{\sin^2 s}{s^2} + \delta\left(\frac{s}{\pi}\right),$$

$s = \pi\omega\nu$ in I, II, III :
agrees with Gaussian Unitary Ensemble (GUE)

IV: **Poisson statistics**

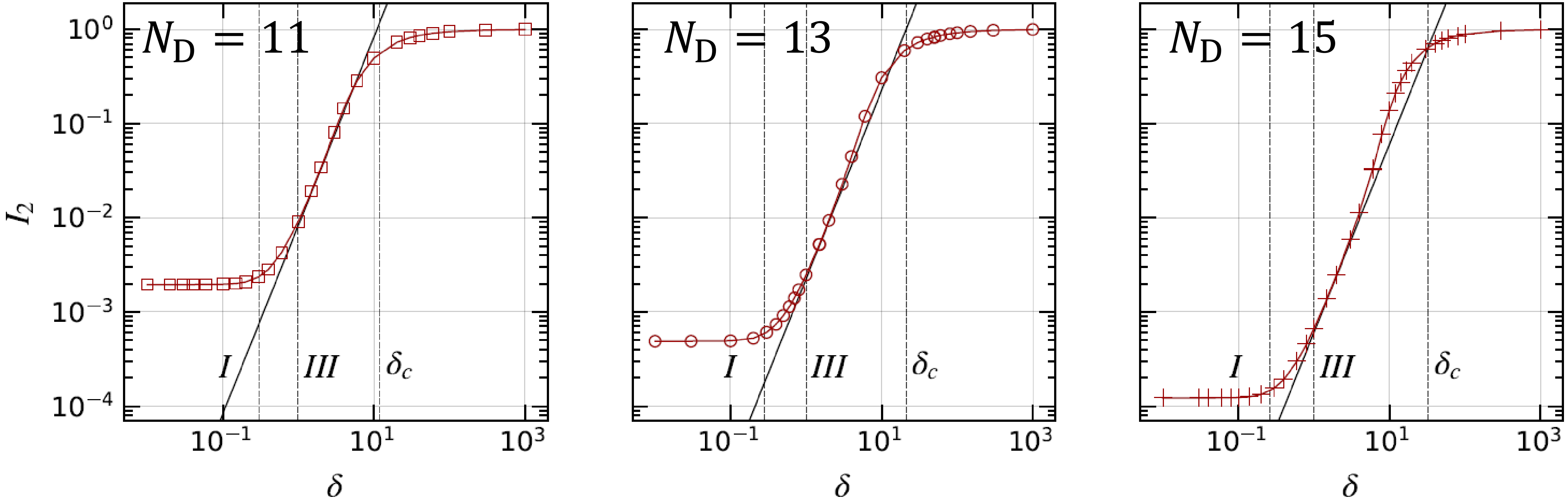
PRR 3, 013023 (2021)

Inverse participation ratio vs prediction for III

IPR $I_2 =$ average of $\sum_n |\langle\psi|n\rangle|^4$ for normalized ψ, $\frac{1}{D} \leq I_2 \leq 1$

Equal weights

Single non-zero element



$$I_q = \frac{q(2q-3)\,!!}{\delta^{2(1-q)}}\left(\frac{\pi D}{4\sqrt{N_{\mathrm{D}}}}\right)^{1-q} = q(2q-3)\,!!\,\left(\frac{4\sqrt{N_{\mathrm{D}}}\delta^2}{2^{N-1}\pi}\right)^{q-1} \text{ in III}$$

Central 1/7 of the energy spectrum

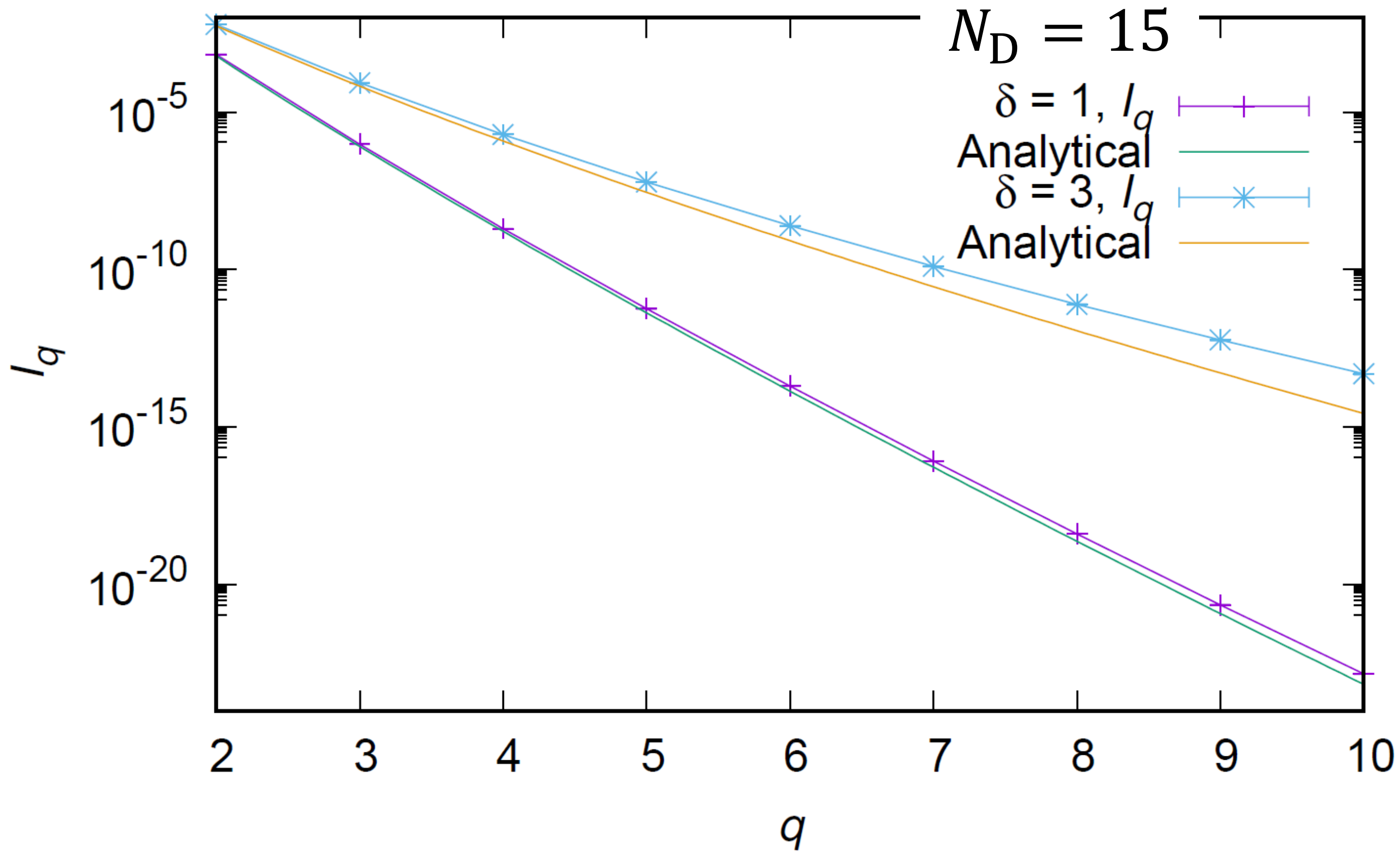
Higher moments of eigenvectors

PRR 3, 013023 (2021)

Analytical prediction:

$$I_q = \frac{q(2q-3)\,!!}{\delta^{2(1-q)}}\left(\frac{\pi D}{4\sqrt{N_{\mathrm{D}}}}\right)^{1-q} = q(2q-3)\,!!\left(\frac{4\sqrt{N_{\mathrm{D}}}\delta^2}{2^{N-1}\pi}\right)^{q-1} \text{ in III}$$

$$I_q = \nu^{-1}\sum_{n,\psi}\langle|\langle\psi|n\rangle|^{2q}\delta(E_\psi)\rangle_J$$



Good agreement up to large q for $\delta \sim 1$

Central 1/7 of the energy spectrum

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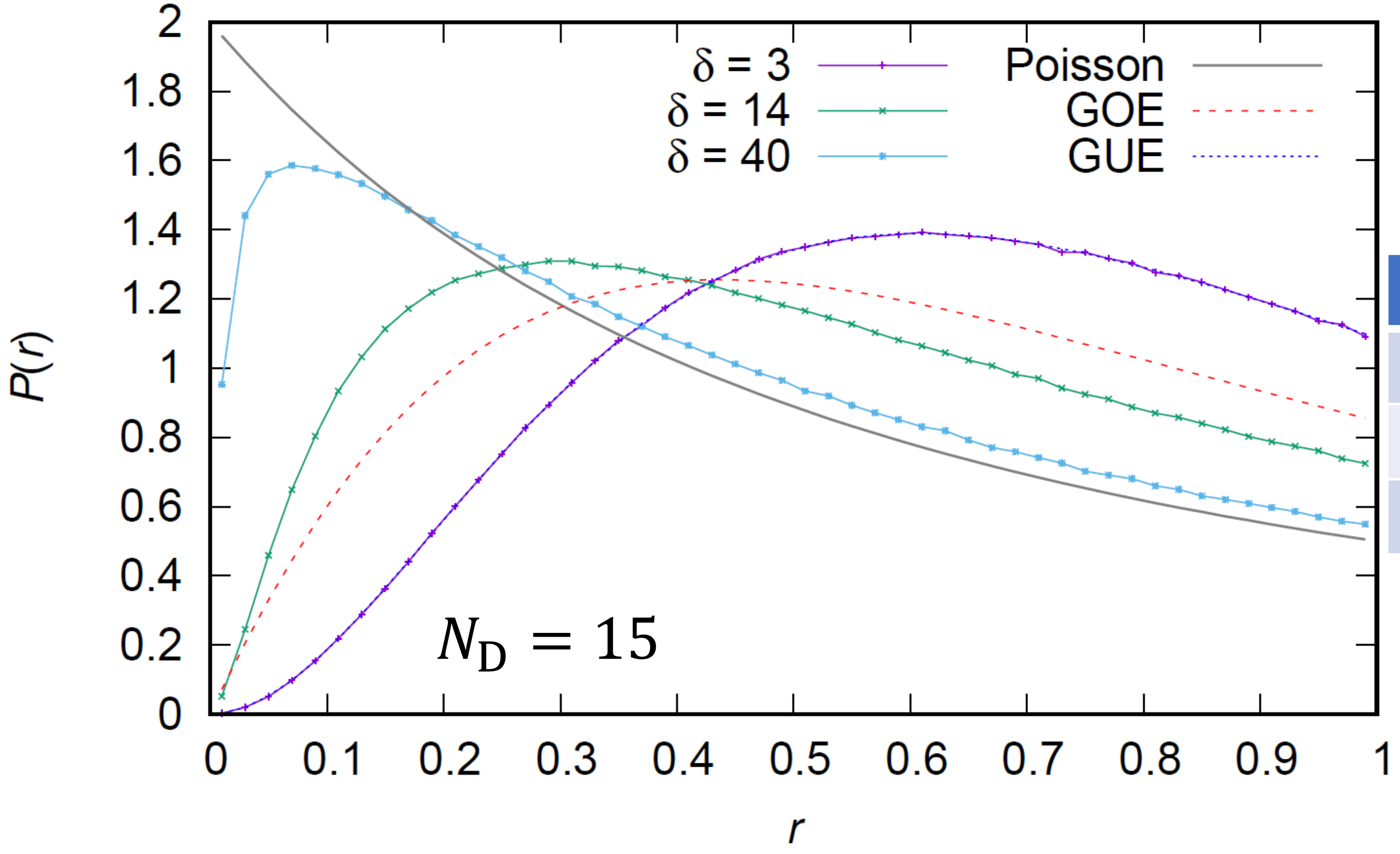
Spectral statistics: gap ratio distribution



(Analytical prediction: $\delta_\mathrm{c} = \frac{Z}{\sqrt{2\rho}} W(2Z\sqrt{\pi}) = 38.47$)

Measure difference by Kullback-Leibler (KL) divergence: $D_\mathrm{KL}(P||Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)}$.

$\boldsymbol{\delta}$	$D_\mathrm{KL}(P(\delta,r)\|\|P_\mathrm{Poisson}(r))$	$D_\mathrm{KL}(P(\delta,r)\|\|P_\mathrm{GUE}(r))$
3	0.3608	5×10^{-6}
14	0.1234	0.1463
40	0.0096	0.5705

$$r = \frac{\min(E_{i+1} - E_i\,,\ \ E_{i+2} - E_{i+1})}{\max(E_{i+1} - E_i\,,\ \ E_{i+2} - E_{i+1})}$$

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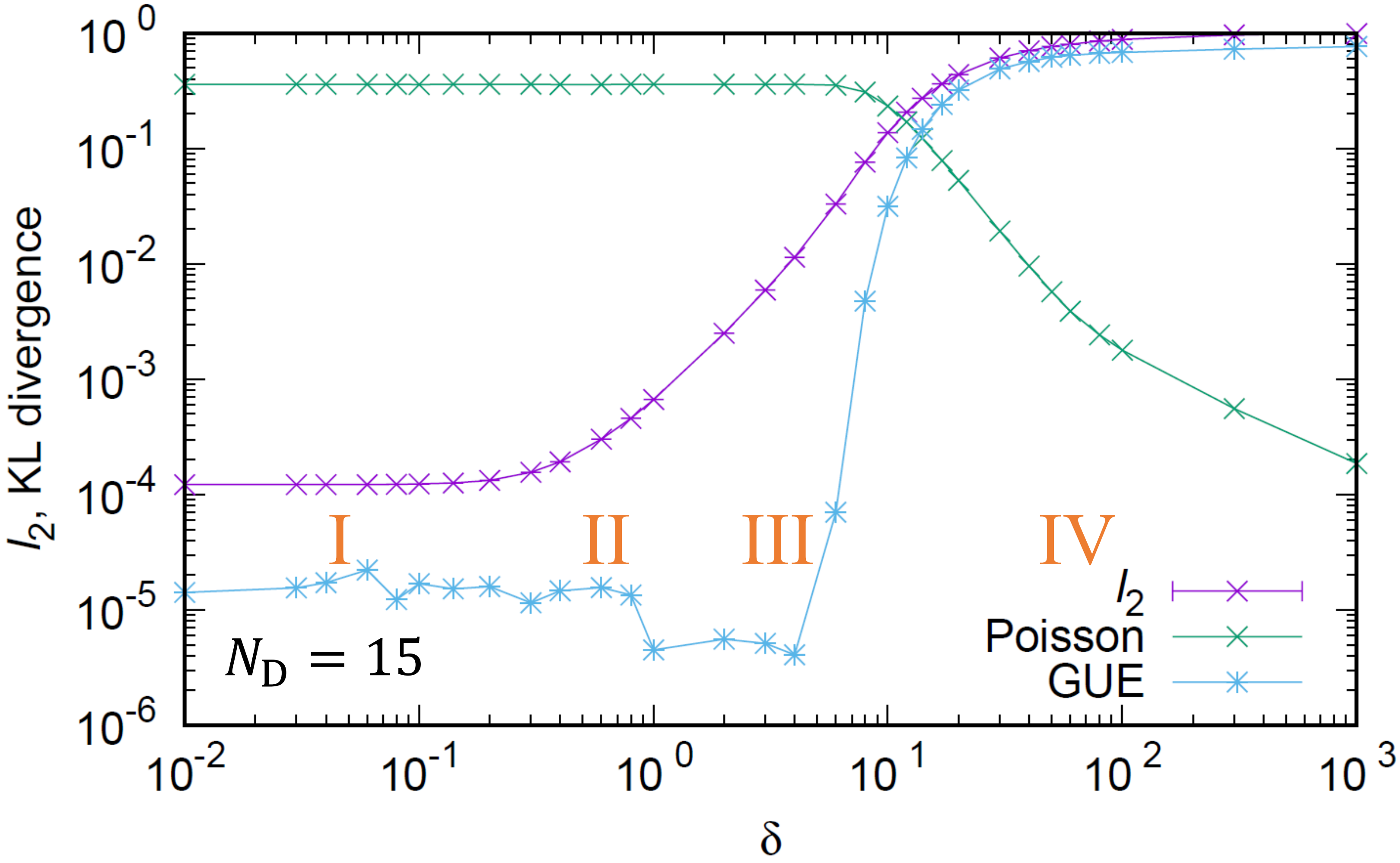
Departure from random matrix $P(r)$ occurs after I_2 has grown significantly

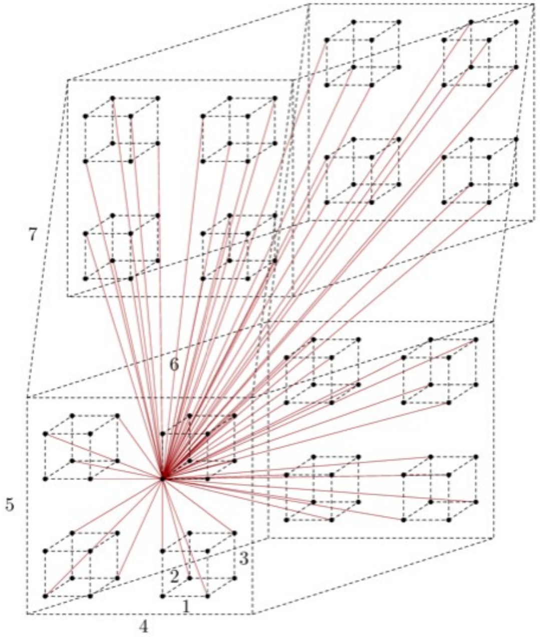


Summary so far…

Felipe Monteiro, Tobias Micklitz, Masaki Tezuka, and Alexander Altland, Phys. Rev. Research 3, 013023 (2021) arXiv:2005.12809

Fock space localization in many-body quantum systems

Sachdev-Ye-Kitaev model as tractable system



Analytical estimate of inverse participation ratio, spectral statistics

Numerical calculation of inverse participation ratio, energy spectrum correlation

Four regimes (I: ergodic, II: localization starts, III: localization rapidly progresses, IV: MBL) found in SYK_4 + δ SYK_2 system (in SYK_2-diagonal basis);
I, II, III are chaotic while IV is not

Prediction for momenta of eigenstate wavefunctions I_q is verified by **parameter free comparison**, and **energy spectrum statistics** is consistent with GUE/Poisson transition well after entering regime III

➔ Are eigenstates ergodic? Behavior of the entanglement entropy?

Physics just outside MBL (regions II & III)?

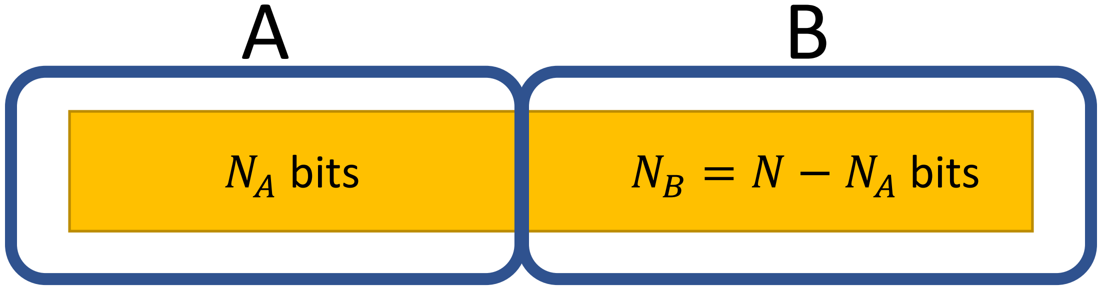
- Thermal phase smoothly connected to extended states (as those in translationally invariant models)?
- Non-ergodic extended (NEE) states discussed for several models
  (Bethe lattice, random regular graphs, disordered Josephson junction chains, …)

“golf course” potential energy landscape

“Non-ergodic extended phase of the Quantum Random Energy Model”
[L. Faoro, M. V. Feigel’man, L. Ioffe, Ann. Phys. **409**, 167916 (2019)]

PRL 127, 030601 (2021)

Evaluation of entanglement entropy



Zero-energy eigenstate $|\psi\rangle$, density matrix $\rho = |\psi\rangle\langle\psi|$

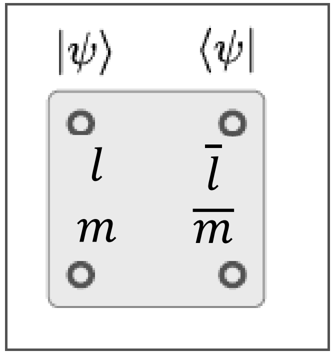
Reduced density matrix $\rho_A = \mathrm{tr}_B \rho$

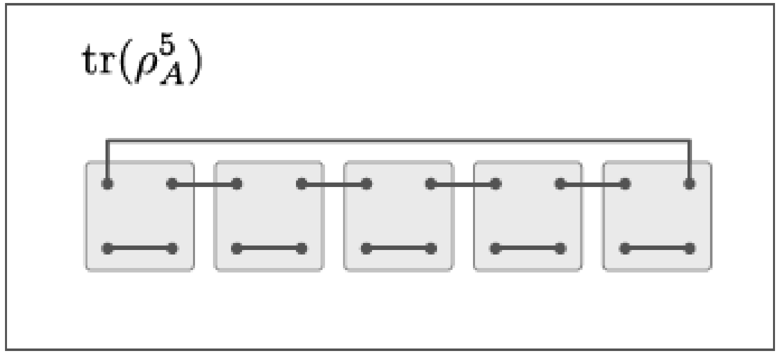
Entanglement entropy $S_A = -\mathrm{tr}_A(\rho_A \ln \rho_A)$

Fock space $\mathcal{F} = \mathcal{F}_A \otimes \mathcal{F}_B$

$$n = (l, m)$$

Evaluate disorder averaged moments $M_r = \langle \mathrm{tr}_A(\rho_A^r)\rangle$, $S_A = -\partial_r M_r|_{r=1}$.





$\mathcal{N} = (n^1, n^2, \dots, n^r), \mathcal{N}_A = (l^1, l^2, \dots, l^r), \mathcal{N}_B = (m^1, m^2, \dots, m^r)$

$$\rho_A^r = \sum_{\substack{l^1,\dots,l^r \\ m^1,\dots,m^r}} \psi^{(l^1,m^1)}\, \overline{\psi}^{(l^2,m^1)} \psi^{(l^2,m^2)}\, \overline{\psi}^{(l^3,m^2)} \cdots \psi^{(l^r,m^r)}\, \overline{\psi}^{(l^1,m^r)}$$

PRL 127, 030601 (2021)

Evaluation of power of reduced density matrix

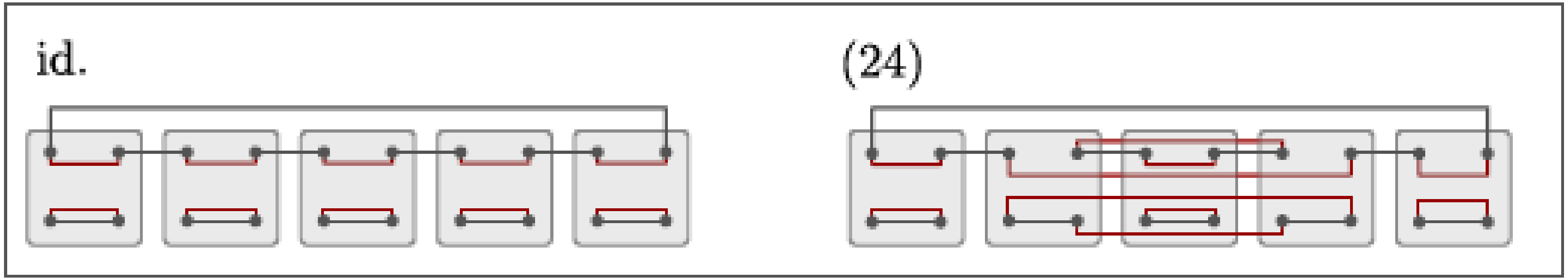
$$\qquad\qquad n^1 \qquad\quad \overline{n}^1 \qquad n^2 \qquad\quad \overline{n}^2$$

$$\rho_A^r = \sum_{\substack{l^1,\dots,l^r \\ m^1,\dots,m^r}} \psi^{(l^1,m^1)}\,\overline{\psi}^{(l^2,m^1)}\,\psi^{(l^2,m^2)}\,\overline{\psi}^{(l^3,m^2)}\cdots\psi^{(l^r,m^r)}\,\overline{\psi}^{(l^1,m^r)}$$

For this sum to survive disorder averaging,

$\mathcal{N} = (n^1, n^2, \dots, n^r)$ and $\overline{\mathcal{N}} = \left(\overline{n}^1, \overline{n}^2, \dots, \overline{n}^r\right)$ should be equal as sets,

$\mathcal{N}^i = \overline{\mathcal{N}}^{\sigma(i)}$



$n^1 = \overline{n}^1, n^2 = \overline{n}^2, n^3 = \overline{n}^3, n^4 = \overline{n}^4, n^5 = \overline{n}^5$

$n^1 = \overline{n}^1, \boldsymbol{n^2 = \overline{n}^4}, n^3 = \overline{n}^3, \boldsymbol{n^4 = \overline{n}^2}, n^5 = \overline{n}^5$

$$M_r = \langle \mathrm{tr}_A(\rho_A^r)\rangle = \sum_{\sigma}\sum_{\mathcal{N}}\prod_{i=1}^{r}\left\langle\left|\psi_{n^i}\right|^2\right\rangle \delta_{\mathcal{N}_A,(\sigma\circ\tau)\mathcal{N}_A}\,\delta_{\mathcal{N}_B,\sigma\mathcal{N}_B}$$

Four regimes of $\mathrm{SYK}_4 + \delta\mathrm{SYK}_2$

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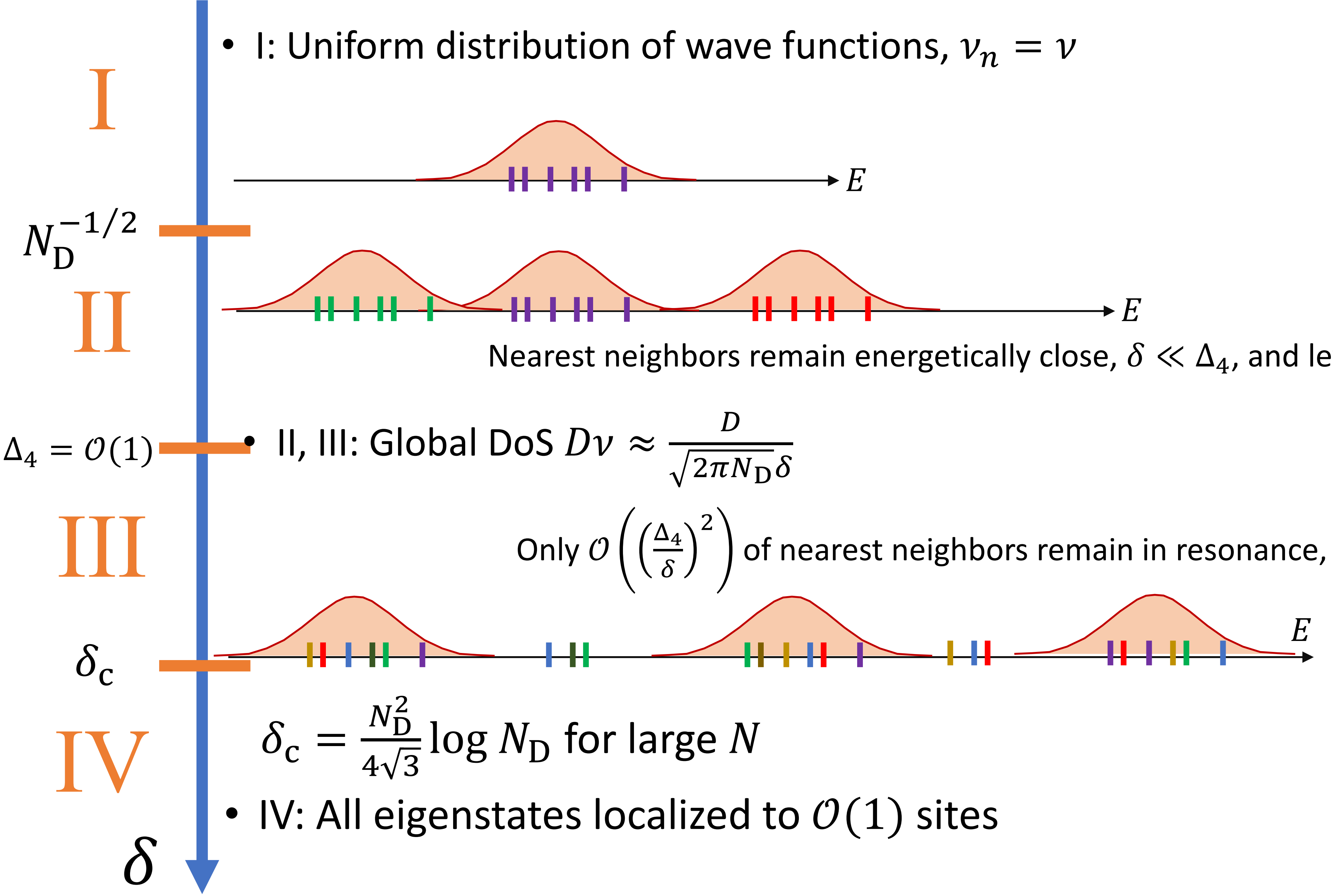

- I: Uniform distribution of wave functions, $\nu_n = \nu$

Nearest neighbors remain energetically close, $\delta \ll \Delta_4$, and level broadening $\kappa = \Delta_4$

- II, III: Global DoS $D\nu \approx \frac{D}{\sqrt{2\pi N_\mathrm{D}}\delta}$

Only $\mathcal{O}\left(\left(\frac{\Delta_4}{\delta}\right)^2\right)$ of nearest neighbors remain in resonance, broadening reduced to $\kappa \sim \Delta_4^2/\delta$

$\delta_\mathrm{c} = \frac{N_\mathrm{D}^2}{4\sqrt{3}} \log N_\mathrm{D}$ for large N

- IV: All eigenstates localized to $\mathcal{O}(1)$ sites

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Regime I: maximally random case

$D_{A(B)} = 2^{N_{A(B)}-1}$

$M_r = \langle \mathrm{tr}_A(\rho_A^r)\rangle,\ S_A = -\partial_r M_r|_{r=1}$

Uniform distribution of wave functions, $\nu_n = \nu$

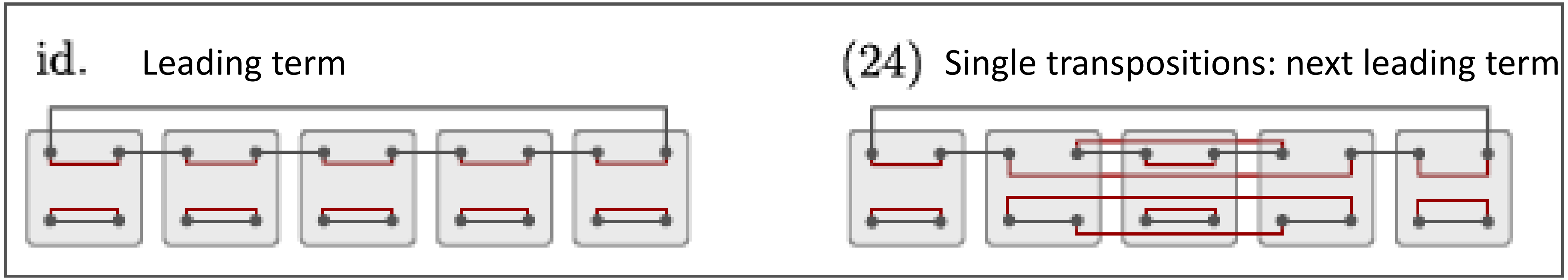
$$M_r \approx D_A^{1-r} + \binom{r}{2} D_A^{2-r} D_B^{-1}$$

Up to single transpositions

Difference from the thermal value $S_{\mathrm{th}} = \ln D_A$

$$\boxed{S_A - S_{\mathrm{th}} = -\frac{D_A}{2D_B}}$$

Exponentially small if $N_A \ll N_B$;
S_A very close to the thermal value



id. Leading term

(24) Single transpositions: next leading term

$$M_r = \langle \mathrm{tr}_A(\rho_A^r)\rangle = \sum_{\sigma}\sum_{\mathcal{N}}\prod_{i=1}^{r} \overset{\boxed{\text{uniform}}}{\left\langle \left|\psi_{n^i}\right|^2 \right\rangle} \delta_{\mathcal{N}_A,(\sigma\circ\tau)\mathcal{N}_A}\,\delta_{\mathcal{N}_B,\sigma\mathcal{N}_B}$$

PRL 127, 030601 (2021)

Regimes II and III: reduced effective dimension

- Assume ergodicity and calculate S_A
- Energy shell: extended cluster of resonant sites (width κ) embedded in the Fock space
- Neighboring sites of n: energy $v_m = v_n \pm \mathcal{O}(\delta)$, much more likely to be in the same shell because $\delta \ll \Delta_2 = \sqrt{N_\mathrm{D}}\delta$

Additional assumptions

- Exponentially large number of sites → self averaging
  (sum over site energies = average over approx. Gaussian distributed contributions of subsystem energies to the total energy)
- Total energy $E \sim E_A + E_B$

➔ Up to single transpositions (justified in $1 \ll N_A \ll N_\mathrm{D}$ & replica limit):

$$S_A - S_\mathrm{th} = -\frac{1}{2}\ln\left(\frac{N_\mathrm{D}}{N_B}\right) + \frac{N_A}{2N_D} - \sqrt{\frac{N_\mathrm{D}}{2N_A}\frac{D_A}{2D_B}}$$

in Regimes II, III ($\frac{1}{\sqrt{N_\mathrm{D}}} \ll \delta < \delta_\mathrm{c} \sim N_\mathrm{D}^2 \ln N_\mathrm{D}$)

$$S_A - S_\mathrm{th} = -\frac{D_A}{2D_B}$$

in Regime I

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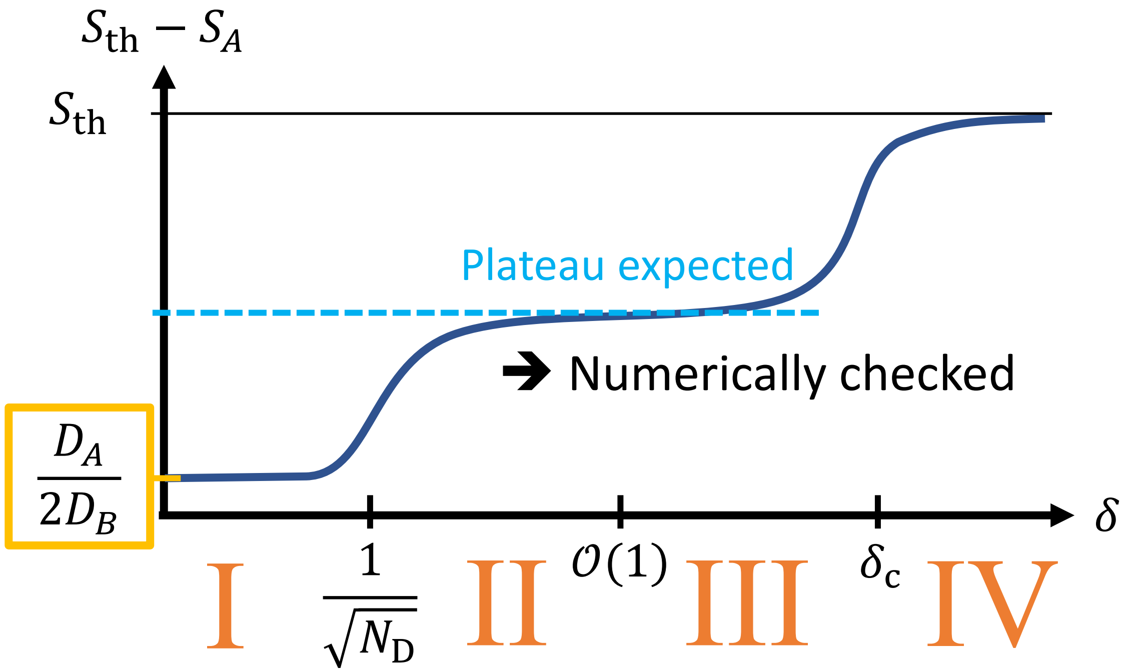
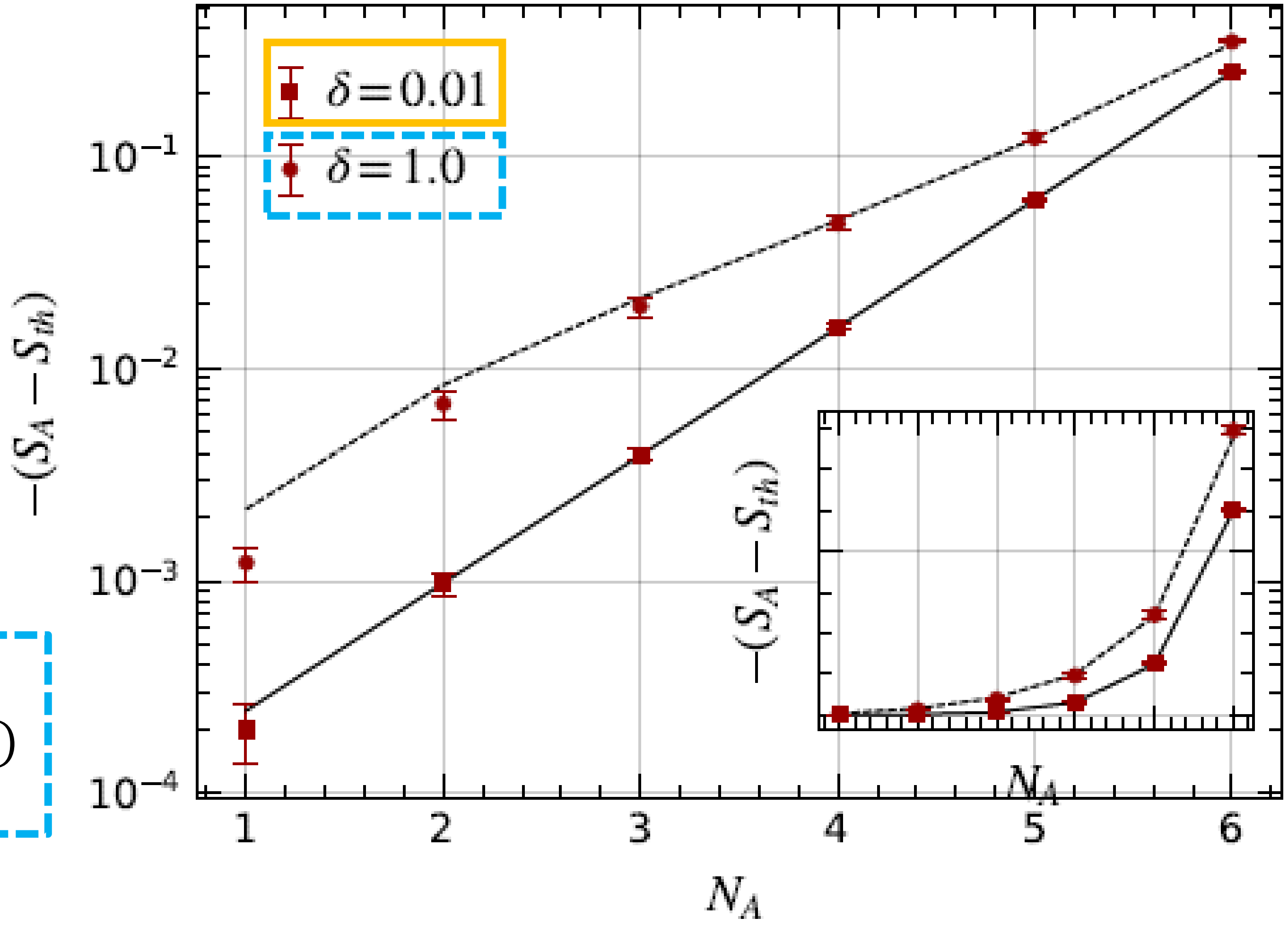
Offset from the thermal value

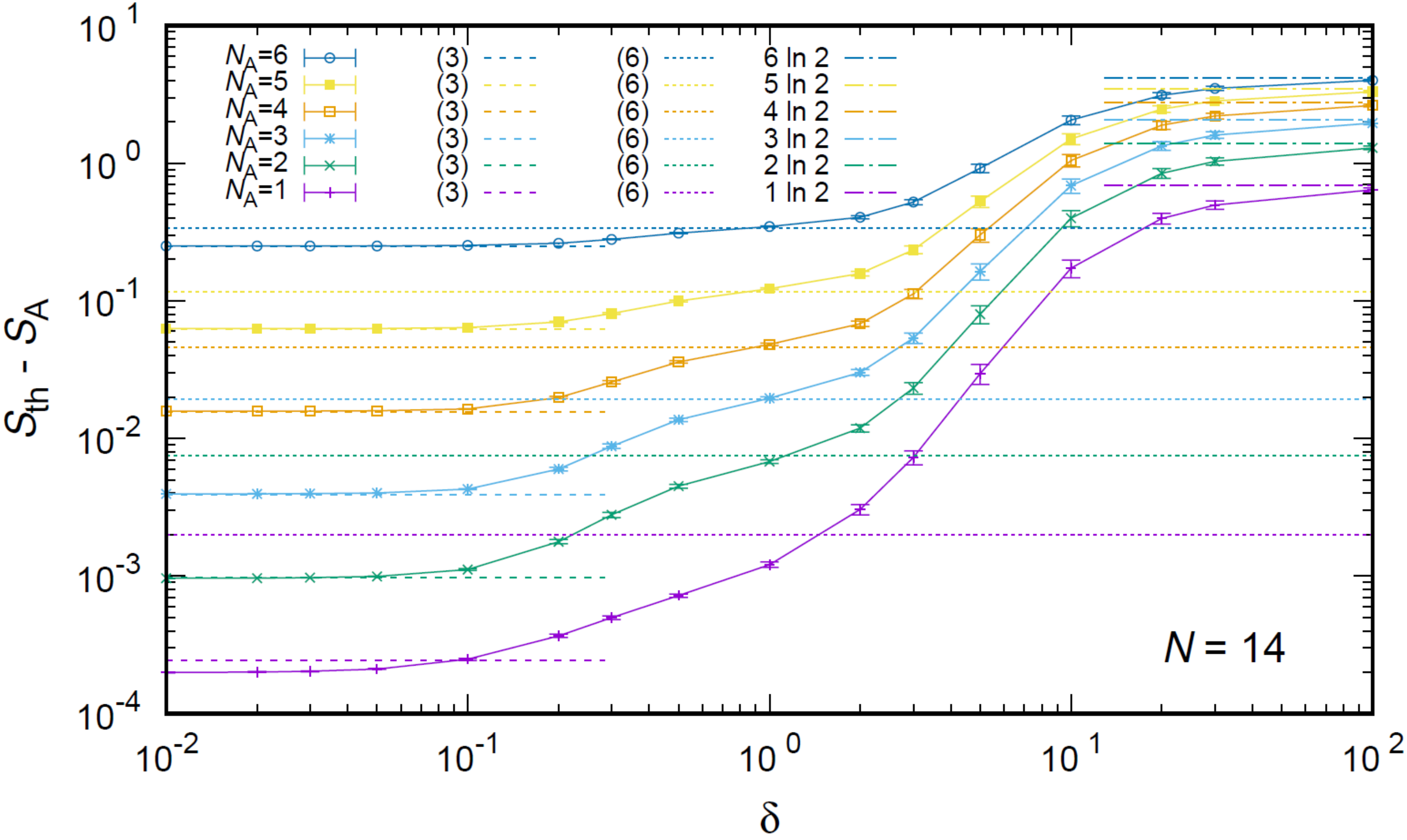


$N_{\rm D} = 14$ ($N = 28$ Majorana fermions)

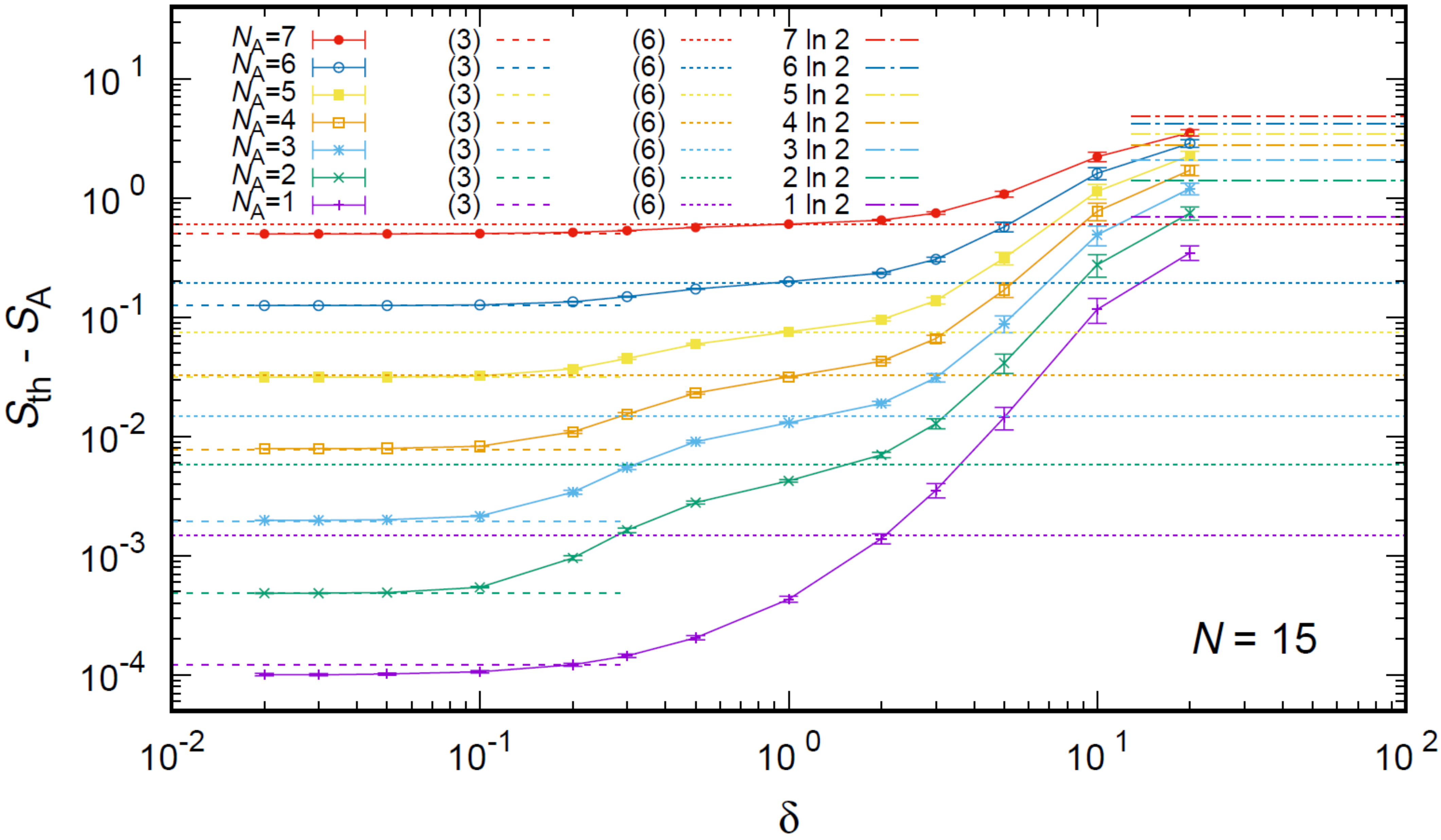


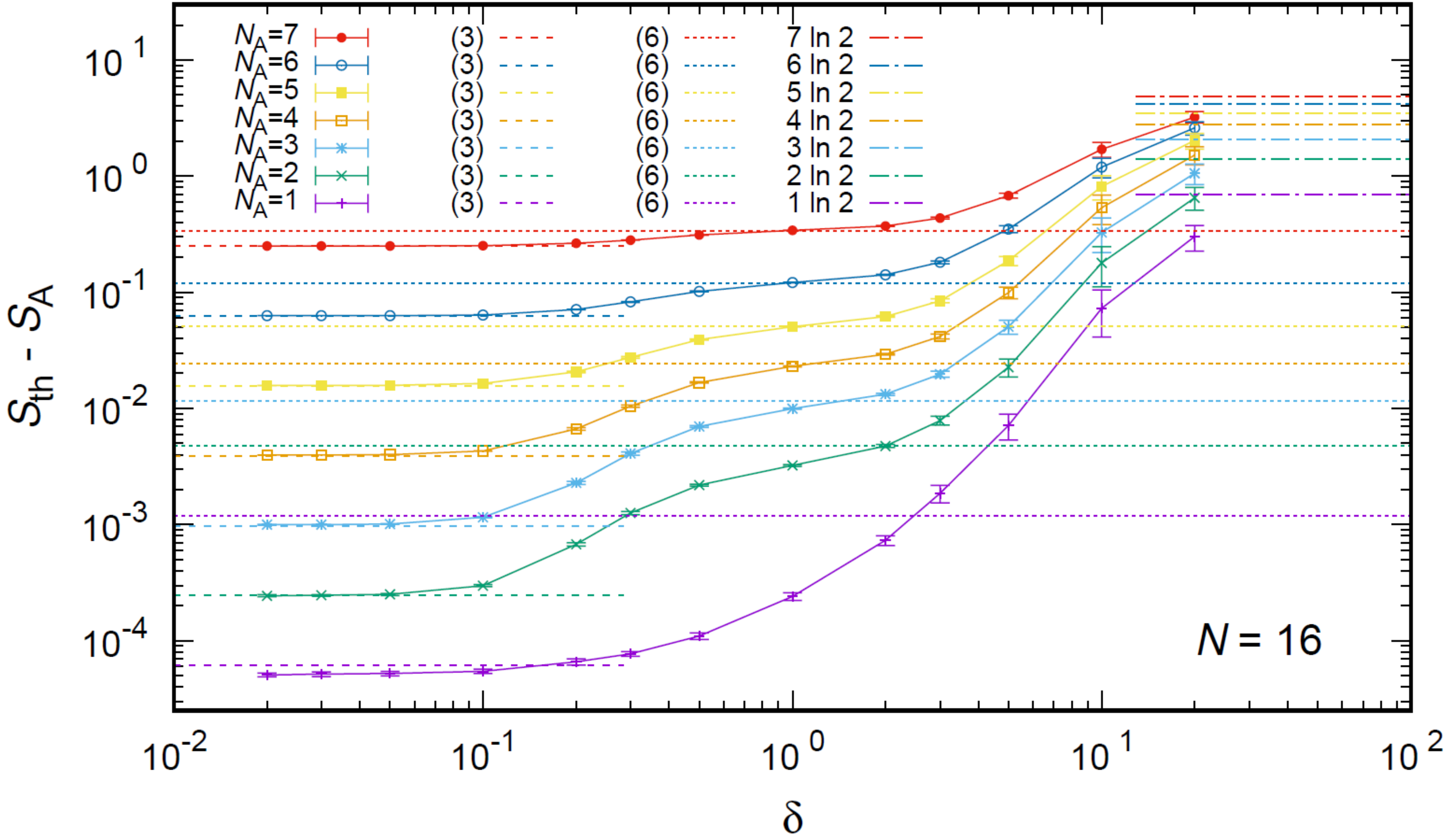
$$S_A - S_{\rm th} = -\frac{1}{2}\ln\left(\frac{N_{\rm D}}{N_B}\right) + \frac{N_A}{2N_D} - \sqrt{\frac{N_{\rm D}}{2N_A}\frac{D_A}{2D_B}} \quad (< 0)$$

in Regimes II, III ($\frac{1}{\sqrt{N_{\rm D}}} \ll \delta < \delta_{\rm c} \sim N_{\rm D}^2 \ln N_{\rm D}$)



N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(3)
(3)
(3)
(3)
(3)
(6)
(6)
(6)
(6)
(6)
(6)
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
$S_{th} - S_A$
δ
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
N = 14


N_A=7
N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(6)
7 ln 2
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
S_{th} - S_A
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
10^0
10^1
10^2
δ
N = 15



N_A=7
N_A=6
N_A=5
N_A=4
N_A=3
N_A=2
N_A=1
(3)
(3)
(3)
(3)
(3)
(3)
(3)
(6)
(6)
(6)
(6)
(6)
(6)
(6)
7 ln 2
6 ln 2
5 ln 2
4 ln 2
3 ln 2
2 ln 2
1 ln 2
S_{th} - S_A
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
δ
N = 16

Phys. Rev. Research 3, 013023 (2021) arXiv:2005.12809

with Felipe Monteiro, Tobias Micklitz, and Alexander Altland

Phys. Rev. Lett. 127, 030601 (2021) arXiv:2012.07884

with F. Monteiro, A. Altland, David A. Huse, and T. Micklitz

Extreme Universe
A New Paradigm for Spacetime and Matter
from Quantum Information

Grant-in-Aid for Transformative Research Areas (A)

Summary

- The Sachdev-Ye-Kitaev (SYK) model: quantum mechanical model realizing chaos bound (~ random matrix, black holes)
- Several experimental proposals, small systems realized
- SYK$_{4+2}$: analytically tractable model for many-body localization (MBL)
  - Fock space: (N/2)-dimensional hypercube
- Analytical results on eigenfunction moments and MBL point

  ➔ Agreement with numerical results without free paramters
- Evaluation of entanglement entropy S_A assuming ergodicity in energy shells

  ➔ Agreement between the numerical and analytical results
- Sparse SYK