Characterization of chaotic dynamics in quantum systems

3rd French Russian Conference on Random Geometry and Physics: Sachdev–Ye–Kitaev Model and Related Topics

Masaki TEZUKA (Kyoto University)

Plan

- Motivation: characterizing quantum dynamics
  - The Sachdev-Ye-Kitaev model as an example
  - From microscopic energy levels
  - By out-of-time-ordered correlators
- Finite-time Lyapunov spectrum in classical chaos
- Quantum Lyapunov spectrum
- Singular value spectrum of two-point correlators
- Summary

1606.02454

1611.04650

1803.08050

1812.04770

1707.02197

1702.06935

1809.01671

1902.11086

Collaborators (in SYK-related papers) and references

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- Pak Hang Chris Lau (MIT→NTHU), Chen-Te Ma (SCNU & Cape Town), Jeff Murugan (Cape Town & KITP)

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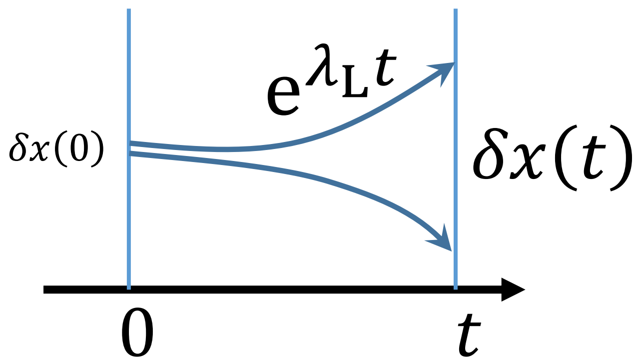
Danshita, Hanada, and MT, PTEP 2017, 083I01 (arXiv:1606.02454)
Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, and MT, JHEP 1705, 118 (2017) (arXiv:1611.04650)
Hanada, Shimada, and MT, Phys. Rev. E **97**, 022224 (2018) (arXiv:1702.06935)
García-García, Loureiro, Romero-Bermudez, and MT, PRL **120**, 241603 (2018) (arXiv:**1707.02197**)
García-García and MT, Phys. Rev. B **99**, 054202 (2019) (arXiv:**1801.03204**)
Gharibyan, Hanada, Shenker, and MT, JHEP 1807, 124 (2018) (arXiv:1803.08050)
Gharibyan, Hanada, Swingle, and MT, JHEP 1904, 082 (2019) (arXiv:**1809.01671**), submitted (arXiv:**1902.11086**)
Lau, Ma, Murugan, Phys. Lett. B in press (arXiv:1812.04770)

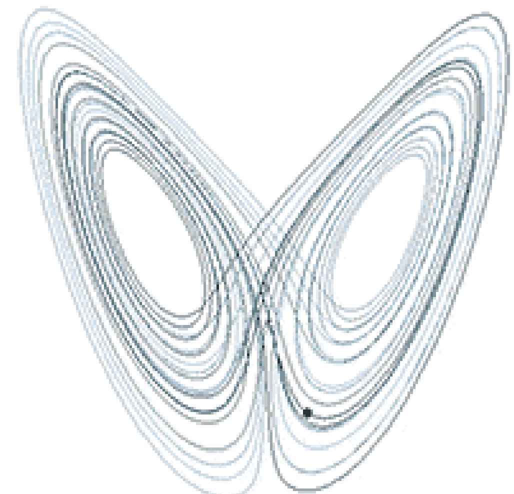
Chaos in deterministic classical dynamics

- Sensitivity to initial conditions: exponential growth of initial perturbation



"butterfly effect"

Bounded, nonperiodic dynamics with nonlinearity
What happens in quantum mechanics?



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(on Wikimedia Commons)

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

VIEW ORIGINAL ARTICLE

Quantamagazine

The Hidden Heroines of Chaos

By Joshua Sokol

May 20, 2019

Two women programmers played a pivotal role in the birth of chaos theory. Their previously untold story illustrates the changing status of computation in science.

Ellen Fetter in 1963, the year Lorenz's seminal paper came out.

Courtesy of Ellen Gille

Building 24, MIT

LEGO Item #21312 “Women of NASA”

Acknowledgments. The writer is indebted to Dr. Barry Saltzman for bringing to his attention the existence of nonperiodic solutions of the convection equations. Special thanks are due to Miss Ellen Fetter for handling the many numerical computations and preparing the graphical presentations of the numerical material.

Acknowledgement

The writer is greatly indebted to Mrs. Margaret Hamilton for her assistance in performing the many numerical computations which were necessary in this work.

How to characterize quantum chaos?

$\hat{H}$ = const.

$$i\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \qquad |\psi(t)\rangle = \hat{\mathrm{T}}\exp\left[-i\int_0^t \hat{H}(t')dt\right]|\psi(t=0)\rangle = \exp\left(-i\hat{H}t\right)|\psi(t=0)\rangle$$

Linear dynamics

Unitary time evolution

- Long time: energy level statistics

Correlation between levels, as in random matrices

Short range: Normalized level separation distribution, gap ratio, …
Longer range: Number variance, spectral form factor, …

cf. Bohigas-Giannoni-Schmit conjecture

- Short time: out-of-time correlator

Classically,

$$\left\{x_i(t), p_j(0)\right\}_{\mathrm{PB}}{}^2 = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)^2 \to e^{2\lambda_{\mathrm{L}}t} \text{ for large } t$$

Quantum version:

OTOC: $C_T(t) = \left\langle\left|\left[\hat{W}(t), \hat{V}(t=0)\right]\right|^2\right\rangle$
$= \left\langle\hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\right\rangle + \cdots$

Numerically

➔ limited to small systems

➔ Hard to see exponential time dependence

The Sachdev-Ye-Kitaev model

N Majorana- or Dirac- fermions randomly coupled to each other

[Majorana version]

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

[Kitaev: talks at KITP (2015)]

[Dirac version]

$$\hat{H} = \frac{1}{(2N)^{3/2}} \sum_{ij;kl} J_{ij;kl} \hat{c}_i{}^{\dagger} \hat{c}_j{}^{\dagger} \hat{c}_k \hat{c}_l$$

[aka two-body random ensemble; French and Wong 1970, Bohigas and Flores 1971, …]
[Kitaev's talk (2015)]
[Sachdev: PRX **5**, 041025 (2015)]

- Solvable in the large *N* limit, Sachdev-Ye "spin liquid" ground state
- Nearly conformal symmetric at low temperature ("emergent …")
- Holographically corresponds to a quantum black hole?
- Experimental schemes have been proposed
- Realizes the Maldacena-Shenker-Stanford chaos bound $\lambda_{\mathrm{L}} = 2\pi k_{\mathrm{B}} T/\hbar$

Generalizations: *q*-fermion interactions "SYK$_q$", supersymmetric SYK, lattice of SYK lands; etc.

A proposal for experimental realization

1606.02454

s: molecular levels

$$\hat{H}_{\mathrm{m}} = \sum_{s=1}^{n_s} \left\{ \nu_s \hat{m}_s^\dagger \hat{m}_s + \sum_{i,j} g_{s,ij} \left(\hat{m}_s^\dagger \hat{c}_i \hat{c}_j - \hat{m}_s \hat{c}_i^\dagger \hat{c}_j^\dagger \right) \right\}.$$

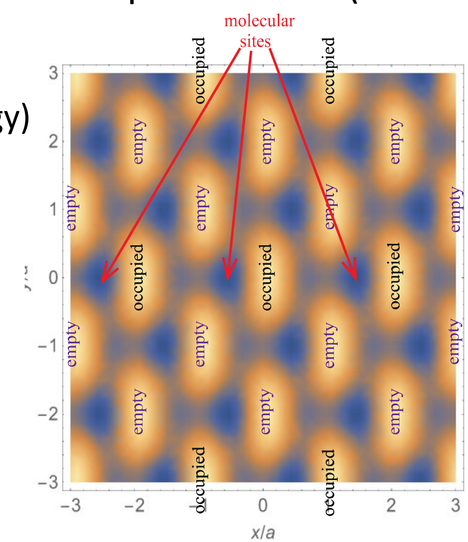
$|\nu_s| \gg |g_{s,ij}|$

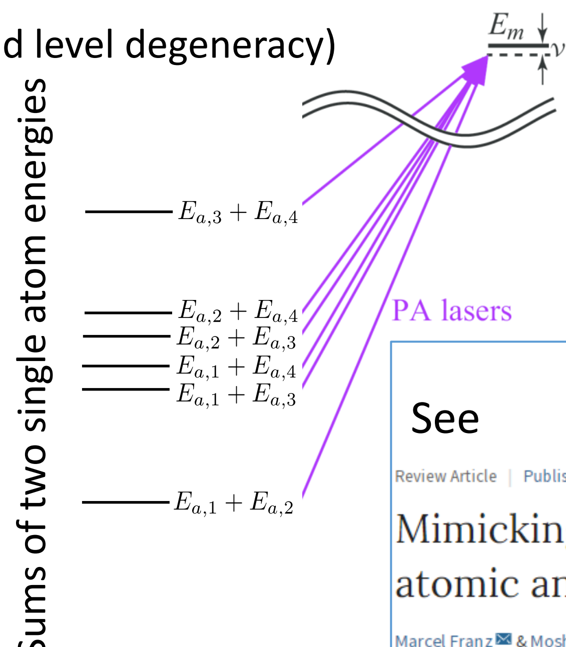
Modified SYK model: $$\hat{H}_{\mathrm{eff}} = \sum_{s,i,j,k,l} \frac{g_{s,ij} g_{s,kl}}{\nu_s} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l.$$

[I. Danshita, M. Hanada, MT: arXiv:1606.02454; PTEP **2017**, 083I01 (2017)]
(also a proceedings manuscript arXiv:1709.07189)

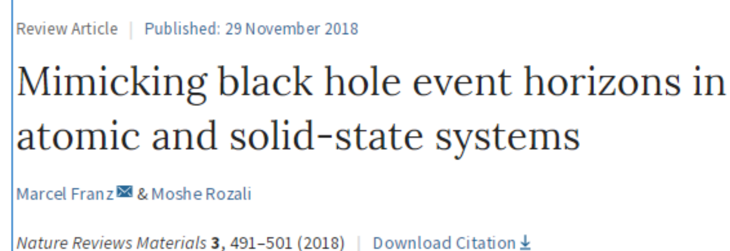
Setup: a double-well optical lattice (no band level degeneracy)

with ^{6}Li
(large recoil energy)





See


Review Article | Published: 29 November 2018

Mimicking black hole event horizons in atomic and solid-state systems

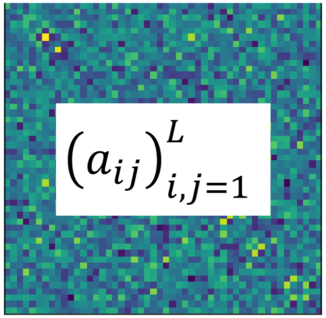
Marcel Franz & Moshe Rozali

Nature Reviews Materials **3**, 491–501 (2018) | Download Citation

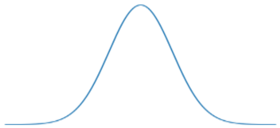
for review including some other proposals e.g. using topological insulator, graphene

cf. Digital Quantum Simulation [García-Álvarez, Egusquiza, Lamata, del Campo, Sonner, and Solano, 1607.08560] PRL 2017

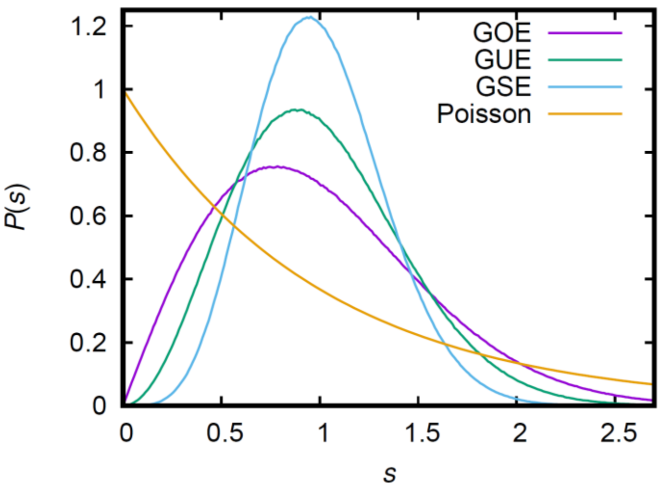
Gaussian random matrices



$a_{ij} = a_{ji}^*$



Gaussian distribution



$$r = \frac{\min(e_{i+1} - e_i\,,\;\; e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\;\; e_{i+2} - e_{i+1})}$$

$$\text{Density} \propto e^{-\frac{\beta K}{4}\mathrm{Tr}H^2} = \exp\left(-\frac{\beta K}{4}\sum_{i,j}^{K}|a_{ij}|^2\right)$$

Real (β=1): Gaussian Orthogonal Ensemble (GOE)

Complex (β=2): G. Unitary E. (GUE)

Quaternion (β=4): G. Symplectic E. (GSE)

Joint distribution function for eigenvalues $\{e_j\}$

Level repulsion

$$p(e_1, e_2, \dots, e_K) \propto \prod_{1\le i<j\le K} |e_i - e_j|^\beta \prod_{i=1}^{K} e^{-\beta K e_i{}^2/4}$$

- $P(s)$: Distribution of normalized level separation $s_j = \frac{e_{j+1} - e_j}{\Delta(\bar{e})}$

  GOE/GUE/GSE: $P(s) \propto s^\beta$ at small s, has e^{-s^2} tail
  Uncorrelated (Poisson): $P(s) = e^{-s}$

- $\langle r \rangle$: Average of neighboring gap ratio

	Uncorrelated	GOE	GUE	GSE
$\langle r \rangle$	2log 2 – 1 = 0.38629…	0.5307(1)	0.5996(1)	0.6744(1)

[Y. Y. Atas *et al.* PRL 2013]

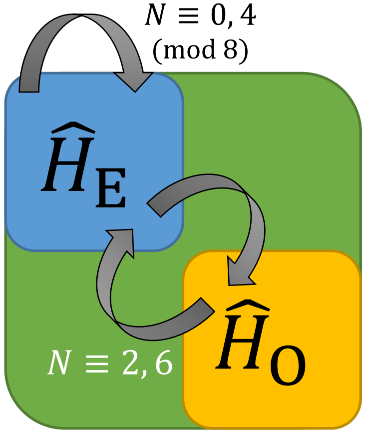
N mod 8 classification of Majorana SYK$_{q=4}$

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\leq a<b<c<d\leq N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

SPT phase classification for class BDI:
$\mathbb{Z} \rightarrow \mathbb{Z}_8$ due to interaction
[L. Fidkowski and A. Kitaev, PRB 2010, PRB 2011]

Introduce $N/2$ complex fermions $\hat{c}_j = \frac{(\hat{\chi}_{2j-1} + \mathrm{i}\hat{\chi}_{2j})}{\sqrt{2}}$

$\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$ respects the complex fermion parity
Even ($\hat{H}_\mathrm{E}$) and odd ($\hat{H}_\mathrm{O}$) sectors: $L = 2^{N/2-1}$ dimensions



N mod 8	0	2	4	6
η	-1	+1	+1	-1
$\hat{X}^2$	**+1**	+1	**-1**	-1
$\hat{X}$ maps H_E to	H_E	H_O	H_E	H_O
Class	**AI**	**A+A**	**AII**	**A+A**
Gaussian ensemble	**GOE**	**GUE**	**GSE**	**GUE**

$$\hat{X} = \hat{K}\prod_{j=1}^{N/2}(\hat{c}_j^\dagger + \hat{c}_j)$$

$$\hat{X}\hat{c}_j\hat{X} = \eta\hat{c}_j^\dagger \quad [\hat{X}, \hat{H}] = 0$$

[You, Ludwig, and Xu, PRB 2017]

[Fadi Sun and Jinwu Ye, 1905.07694] for SYK$_q$, supersymmetric SYK

SYK: sparse matrix, but energy spectral statistics strongly resemble that of the corresponding (dense) Gaussian ensemble [Cotler, …, MT, JHEP 2017]

1611.04650

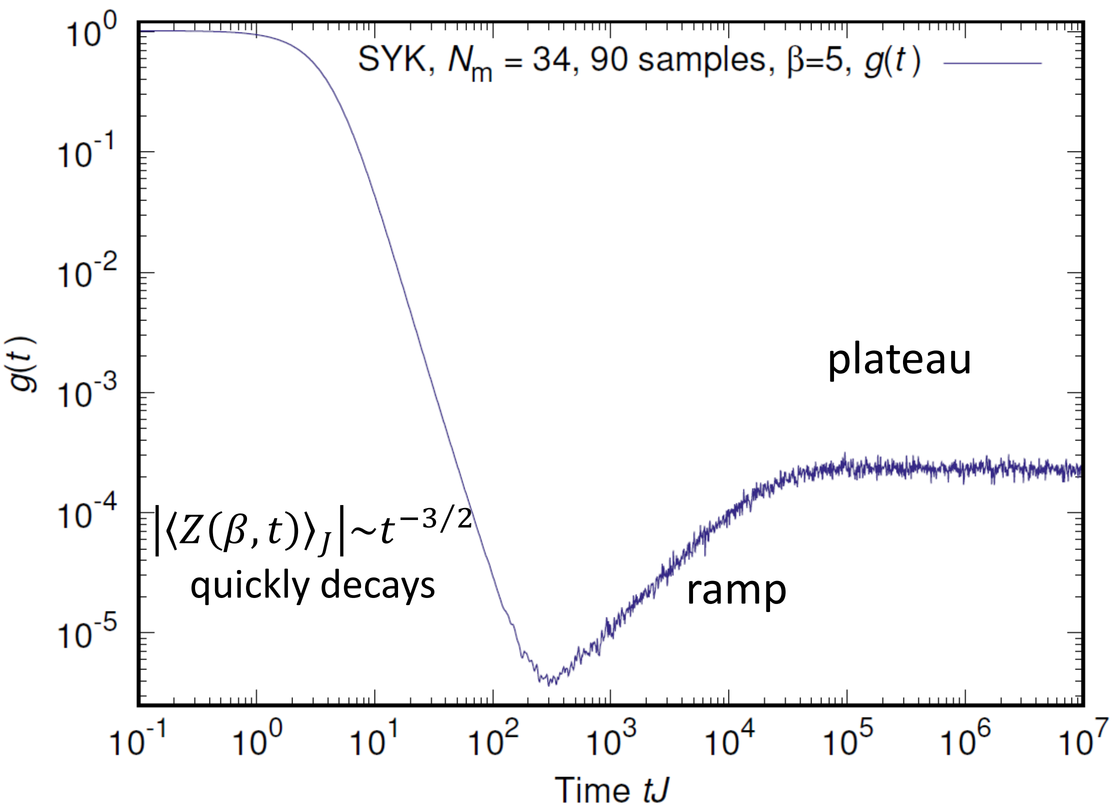
Spectral form factor

Real-time two-point correlation function

$$G(t) = \langle \hat{\chi}_a(t)\hat{\chi}_a(0)\rangle_\beta = \frac{1}{Z(\beta, t=0)}\sum_{m,n} \mathrm{e}^{-\beta E_m}|\langle m|\hat{\chi}_a|n\rangle|^2 \mathrm{e}^{\mathrm{i}(E_m - E_n)t}$$

$$g(\beta, t) = \left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2 = \frac{1}{Z(\beta,t=0)^2}\sum_{m,n} \mathrm{e}^{-\beta(E_m+E_n)}\mathrm{e}^{\mathrm{i}(E_m-E_n)t}$$

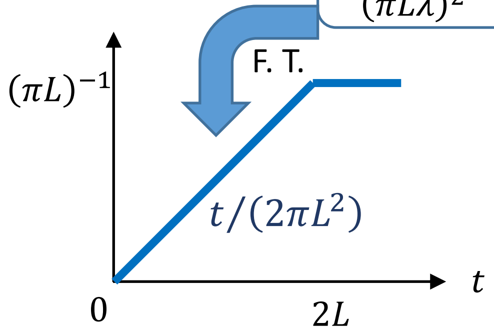
with $Z(\beta,t) = Z(\beta + \mathrm{i}t) = \mathrm{Tr}\left(\mathrm{e}^{-\beta\hat{H} - \mathrm{i}\hat{H}t}\right)$



$$g_c(\beta,t) = \frac{\langle |Z(\beta,t)|^2\rangle_J - |\langle Z(\beta,t)\rangle_J|^2}{\langle Z(\beta)\rangle_J^{\ 2}}$$

$$\sim \iint d\lambda_1 d\lambda_2 \langle \delta\rho(\lambda_1)\delta\rho(\lambda_2)\rangle e^{\mathrm{i}t(\lambda_1 - \lambda_2)}$$

$$R(\lambda) = \langle \delta\rho(\lambda_1)\delta\rho(\lambda_1 - \lambda)\rangle = -\frac{\sin^2 L\lambda}{(\pi L\lambda)^2} + \frac{1}{\pi L}\delta(\lambda)$$



in RMT

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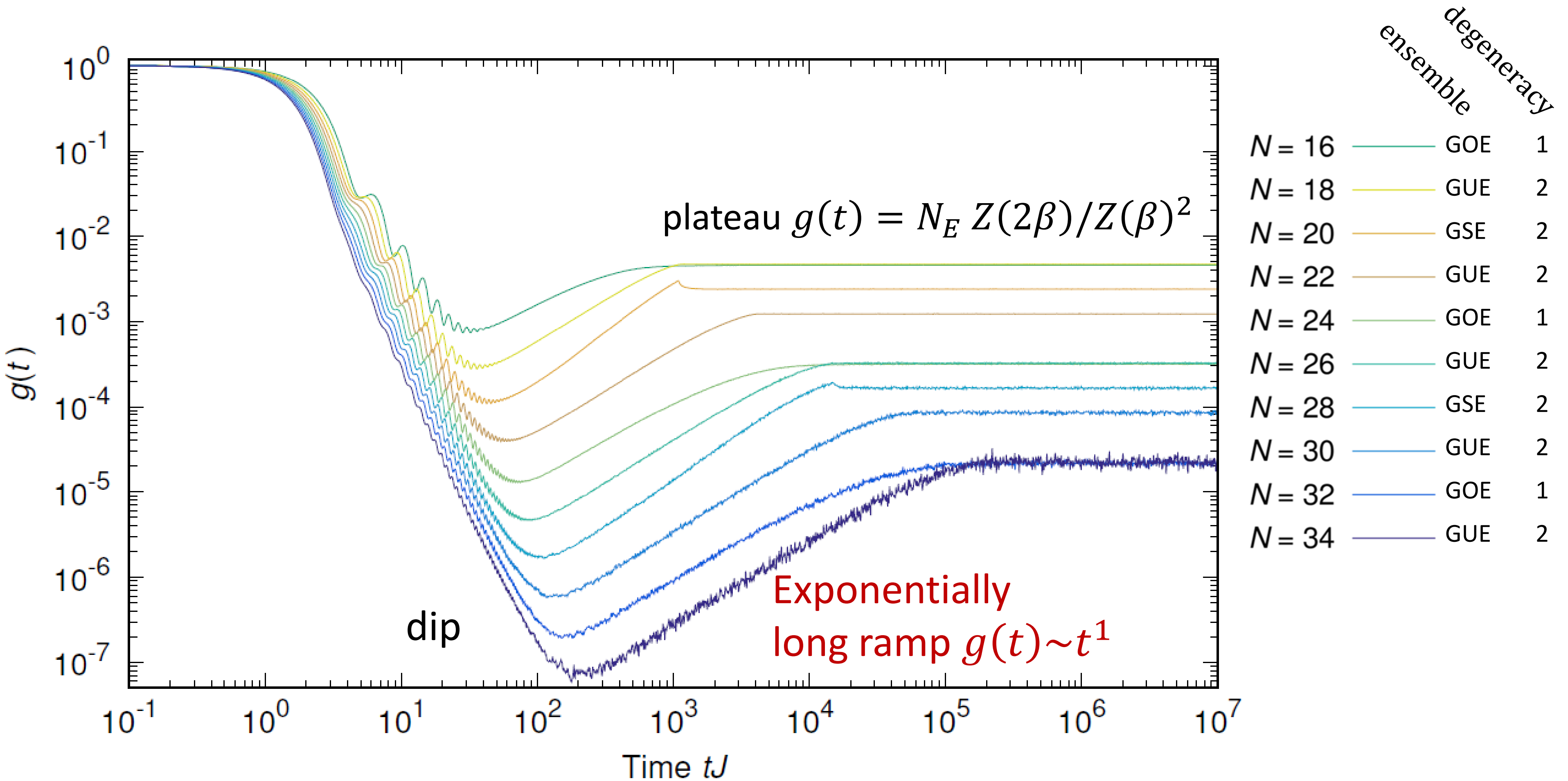
The SYK spectral form factor: *N* dependence

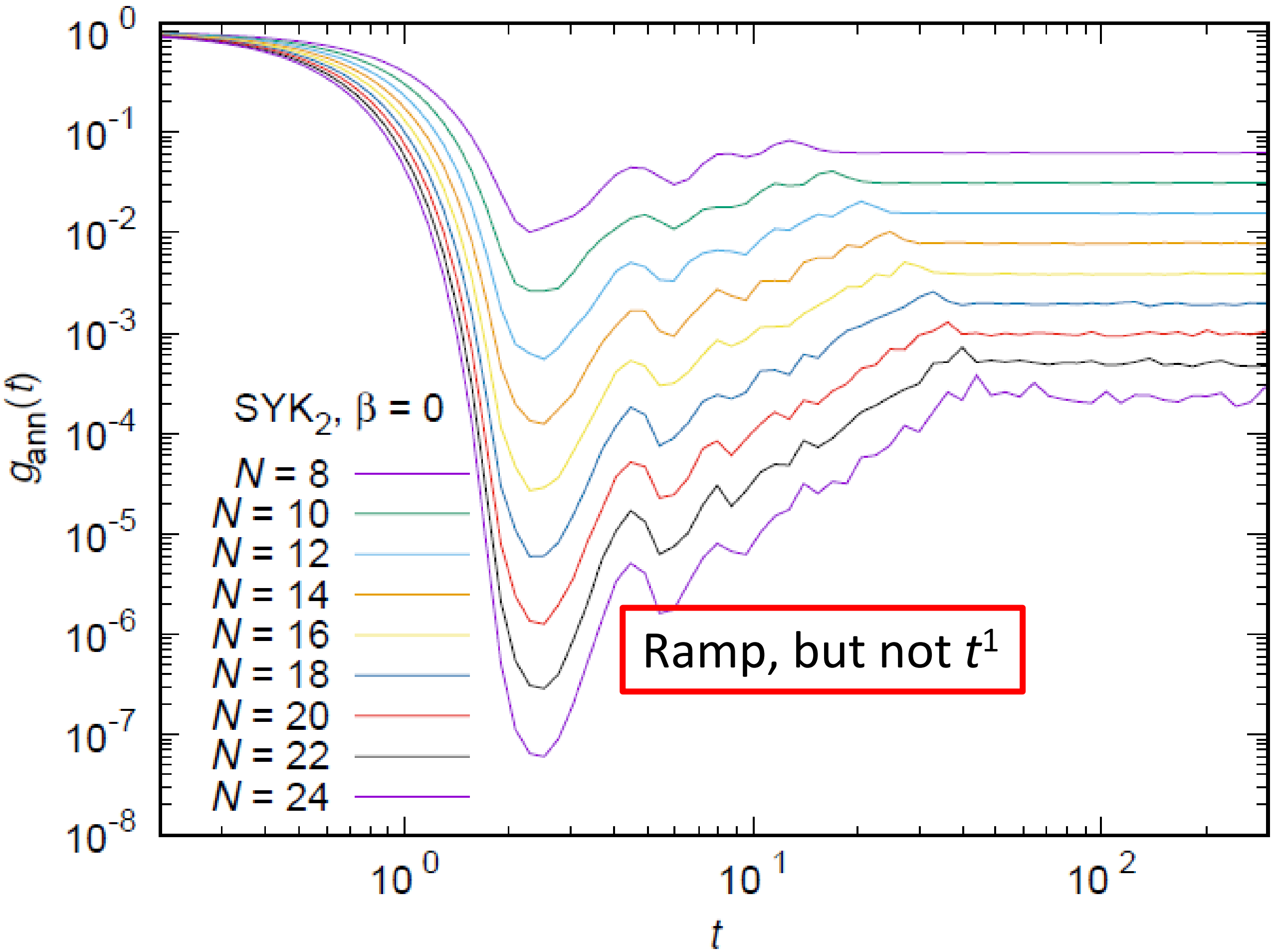


[Cotler, MT et al., JHEP05(2017)118];
for dip time dependence on *N* see [Gharibyan-Hanada-Shenker-MT, JHEP07(2018)124].

1812.04770

Note: dip-ramp-plateau structure does not require chaos

"Randomness and Chaos in Qubit Models"
Pak Hang Chris Lau, Chen-Te Ma,
Jeff Murugan, and MT, Phys. Lett. B in press

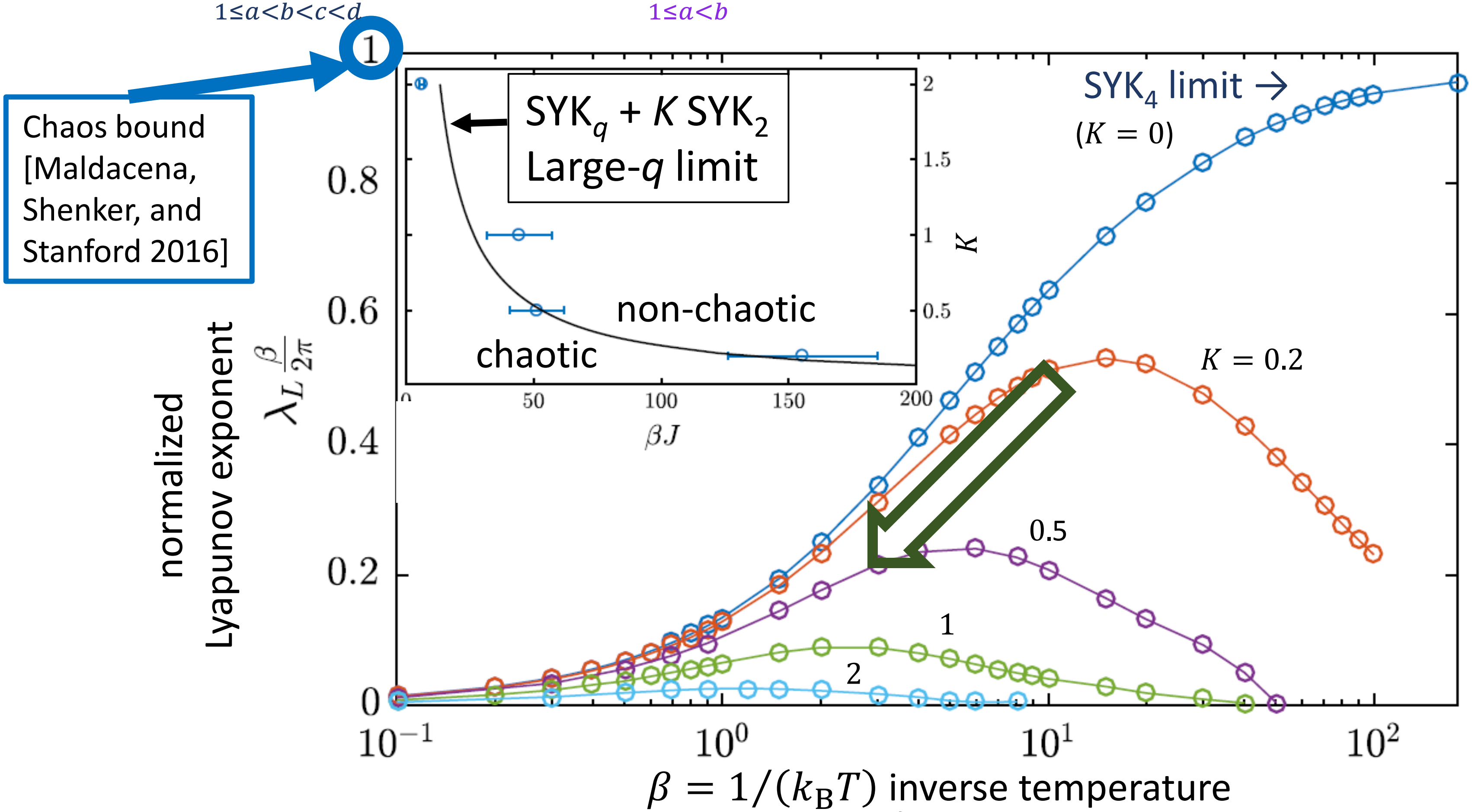


$SYK_4 + SYK_2$: Large-N calculation for OTOC

1707.02197

$$\hat{H} = \underbrace{\sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}_{\text{SYK}_4} + i \underbrace{\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}_{\text{SYK}_2}$$

K_{ab}: standard deviation $\frac{K}{\sqrt{N}}$



A. M. Garcia-Garcia, B. Loureiro, A. Romero-Bermudez, and MT, PRL **120**, 241603 (2018)

Deviation from the chaos bound as SYK_2 component is introduced

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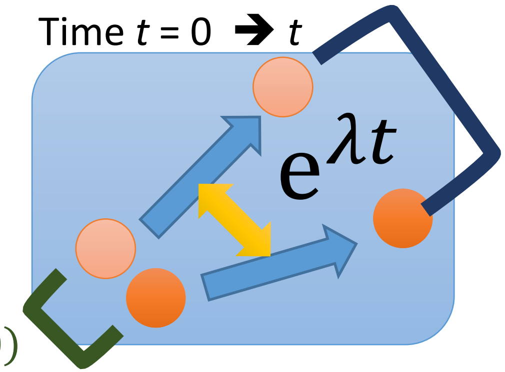
Finite-time Lyapunov spectrum in classical chaotic systems

Classical systems with K degrees of freedom



Deviation at t : linear in initial deviation

$$\delta\phi_i(t) = T_{ij}\delta\phi_j(0)$$

Singular values of T_{ij}: $\{a_k(t)\}_{k=1}^{K}$

Time-dependent Lyapunov spectrum

$$\left\{\lambda_k(t) = \frac{\log a_k(t)}{t}\right\}_{k=1,2,\ldots,K}$$

- Usually we consider the $t \to \infty$ limit, chaotic if $\max(\lambda_k) > 0$
- We focus on finite time behavior

Spectrum $\{\lambda_k(t)\}_{k=1}^{K}$ depends on system details

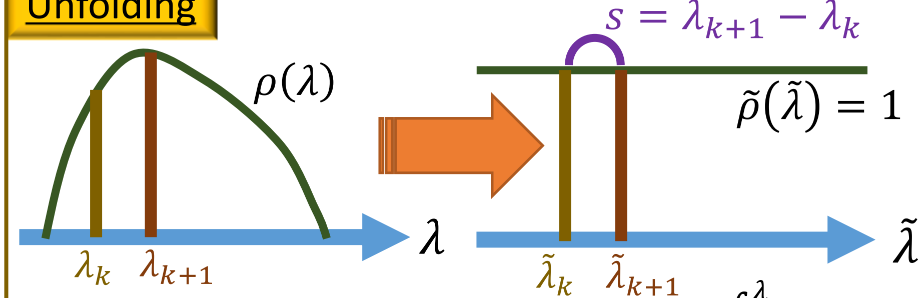
➔ Any universality for chaotic cases?

In many chaotic systems, for large K, the Lyapunov spectrum behaves like that of a Gaussian random matrix at some time scale.

Numerical evidences

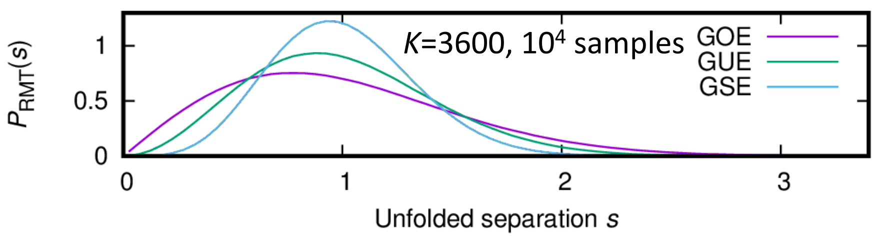
Level separation distribution $P(s)$ for the unfolded Lyapunov spectrum approaches that of random matrix eigenvalues $P_{\mathrm{RMT}}(s)$ at some time scale if the degree of freedom K is large, so does the spectral form factor.

Unfolding



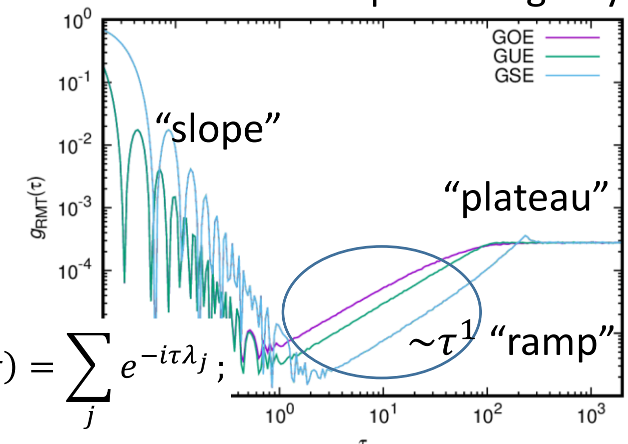
$\rho(\lambda)$: Smoothed density of states

$$\tilde{\lambda} = \int_{-\infty}^{\lambda} \rho(\lambda')\, d\lambda'$$



Spectral form factor

Characterizes the spectral rigidity



$$Z(\tau) = \sum_j e^{-i\tau\lambda_j}\,;$$

$$g(\tau) = \frac{|Z(\tau)|^2}{Z(0)^2} = K^{-2} \sum_{jk} e^{i\tau(\lambda_j - \lambda_k)}$$

Models: logistic map, Lorenz attractor,
D0 brane matrix model (without fermions) and its mass deformation, random band matrix products

cf. Lyapunov spectrum for random coupling [S. K. Patra and A. Ghosh]

- Kuramoto model [PRE **93**, 032208 (2016)],
- Map networks [EPL **117**, 60002 (2017)]

Strong coupling ⇔ GOE
Weak coupling ⇔ Poisson

1702.06935

Example (1): D0 brane matrix model and its deformation

$$L = \frac{N}{2}\mathrm{Tr}\left(\sum_{I=1}^{d=9}(D_t X_I)^2 + \frac{1}{2}\sum_{I\neq J}[X_I, X_J]^2\right) - \frac{Nm^2}{4}\mathrm{Tr}\sum_{I=1}^{d} {X_I}^2$$

- No fermions: classical
- Deformation with "mass" m

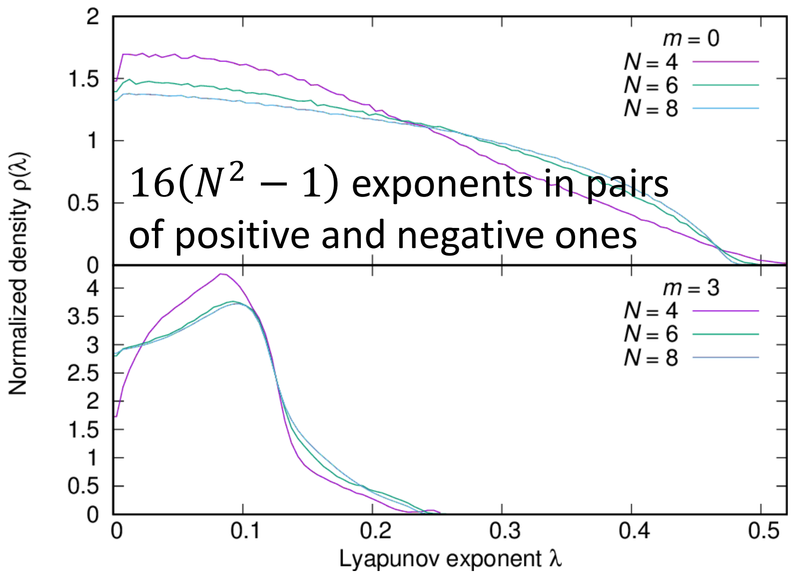
T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997)

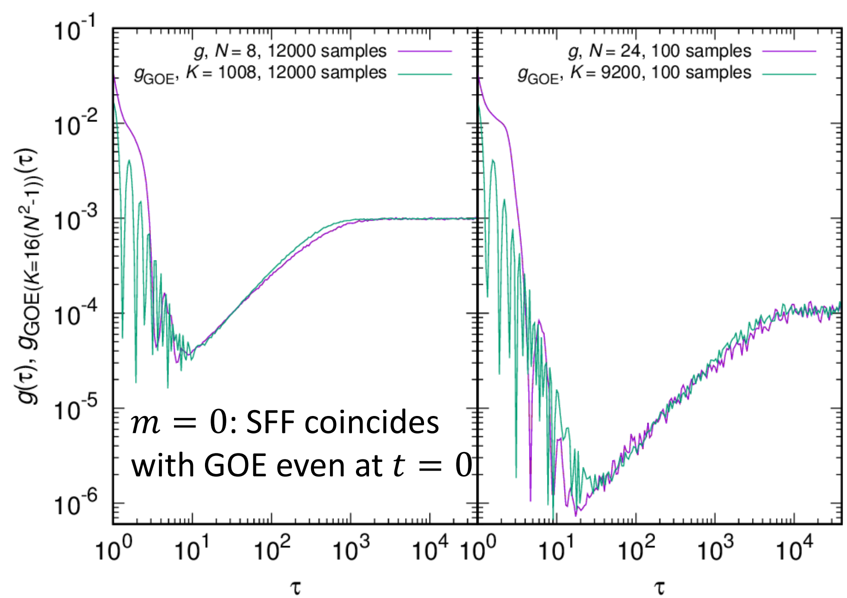
N: matrix dimension (large N limit: strong coupling limit of the type IIA string theory)

X_I: matrices ($16(N^2-1)$ independent entries)

Simulation at constant energy

$$E = 6(N^2-1) - 27$$





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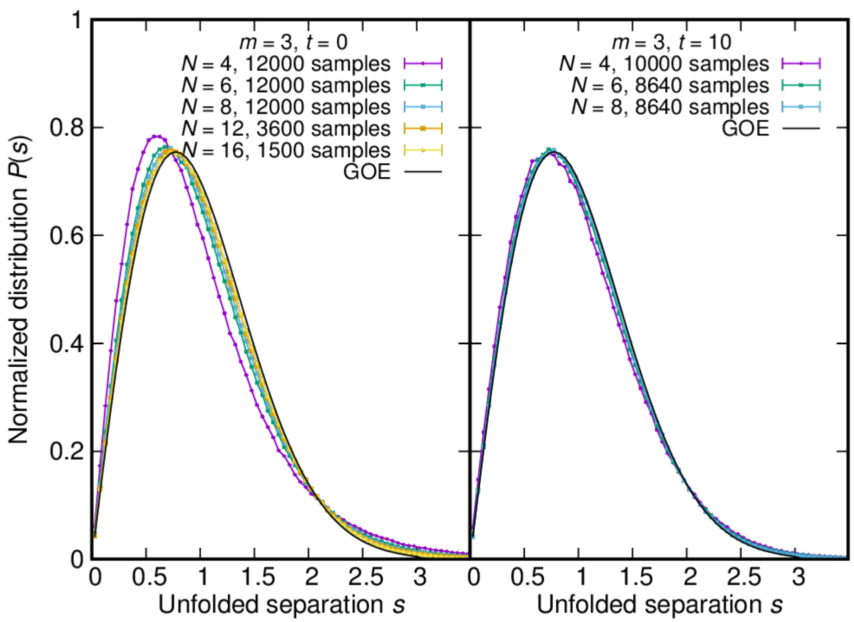
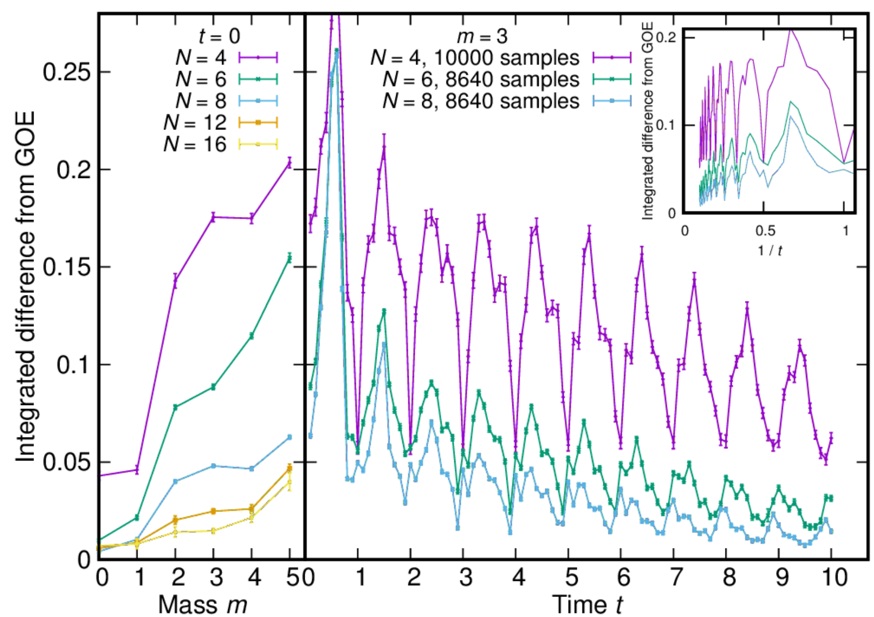
Example (1): D0 brane matrix model and its deformation

$$L = \frac{N}{2}\mathrm{Tr}\left(\sum_{I=1}^{d=9}(D_t X_I)^2 + \frac{1}{2}\sum_{I\neq J}[X_I, X_J]^2\right) - \frac{Nm^2}{4}\mathrm{Tr}\sum_{I=1}^{d} {X_I}^2$$

Time $t = 0$: for $m > 0$, $P(s)$ deviates from RMT



$P(s)$ approaches GOE over time



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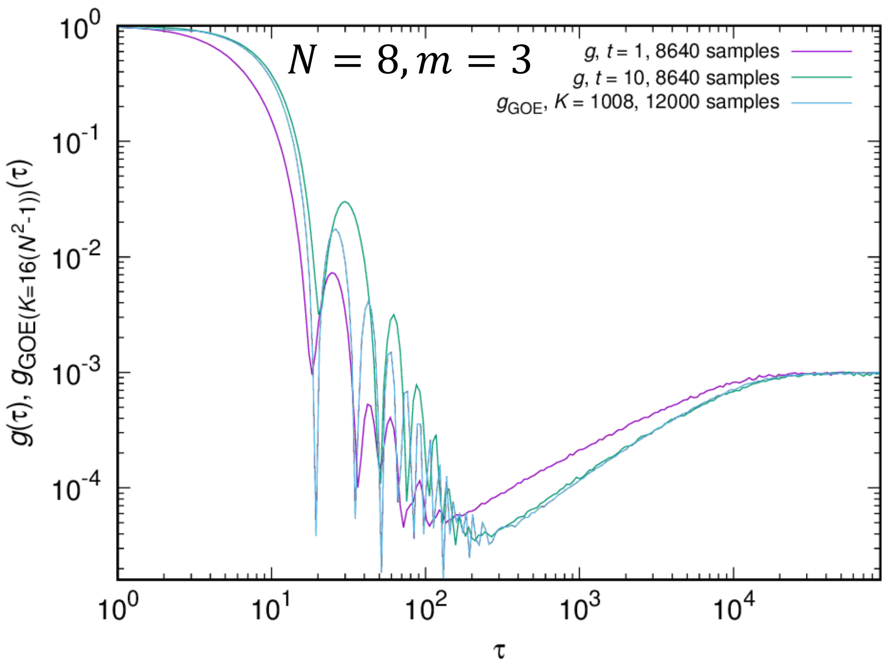
Example (1): D0 brane matrix model and its deformation

$$L = \frac{N}{2}\mathrm{Tr}\left(\sum_{I=1}^{d=9}(D_t X_I)^2 + \frac{1}{2}\sum_{I\neq J}[X_I, X_J]^2\right) - \frac{Nm^2}{4}\mathrm{Tr}\sum_{I=1}^{d} {X_I}^2$$

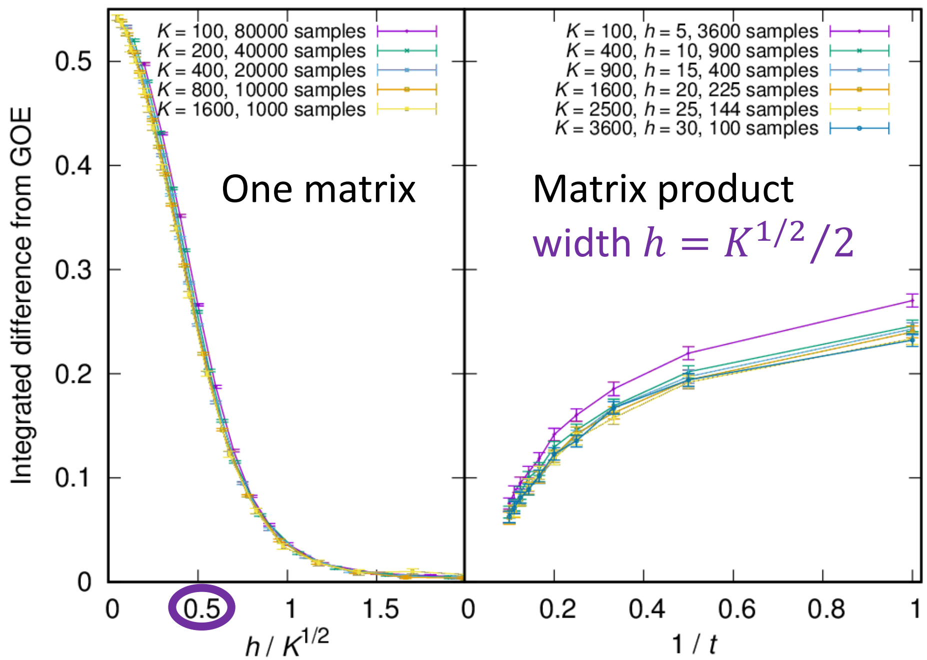
Spectral form factor also approaches GOE

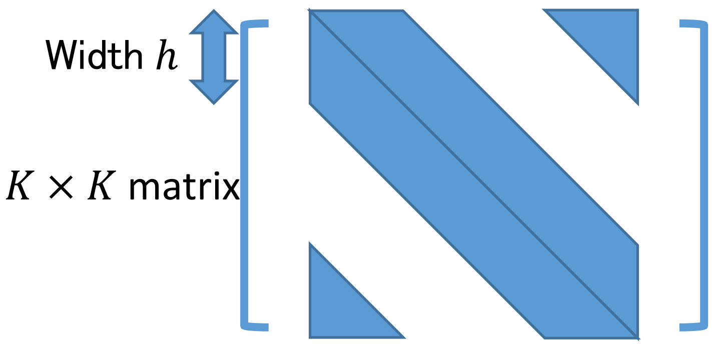


Example (2): Random band matrix product

$$T_{ij} = M^{(1)}_{ii'} M^{(2)}_{i'i''} \cdots M^{(t)}_{j'j} \qquad M^{(l)}_{ij} = 0 \quad \text{if } |i - j \bmod K| \geq h$$

Real random: $P_{\mathrm{GOE}}(s)$ approached
($P_{\mathrm{GUE}}(s)$ if complex random)





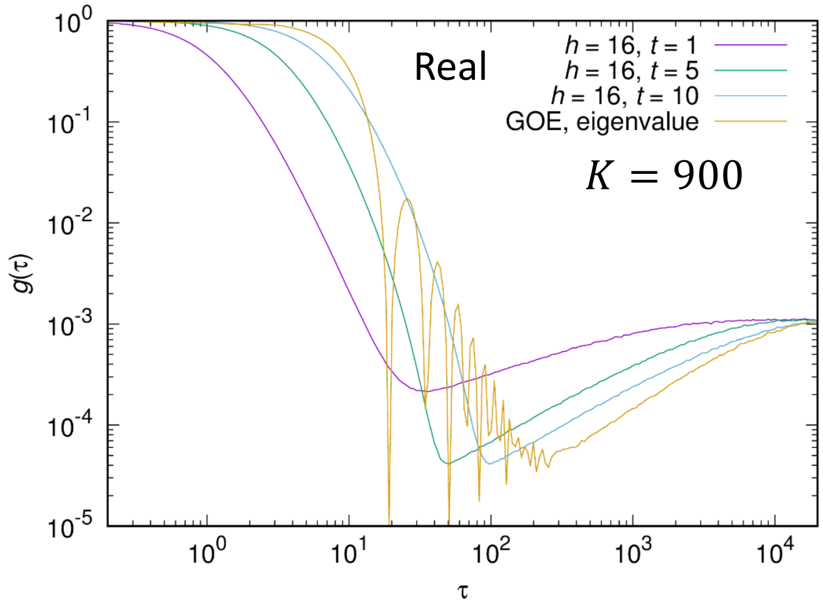
Singular values of T_{ij}: $\{a_k(t)\}_{k=1}^{K}$
Time-dependent Lyapunov spectrum

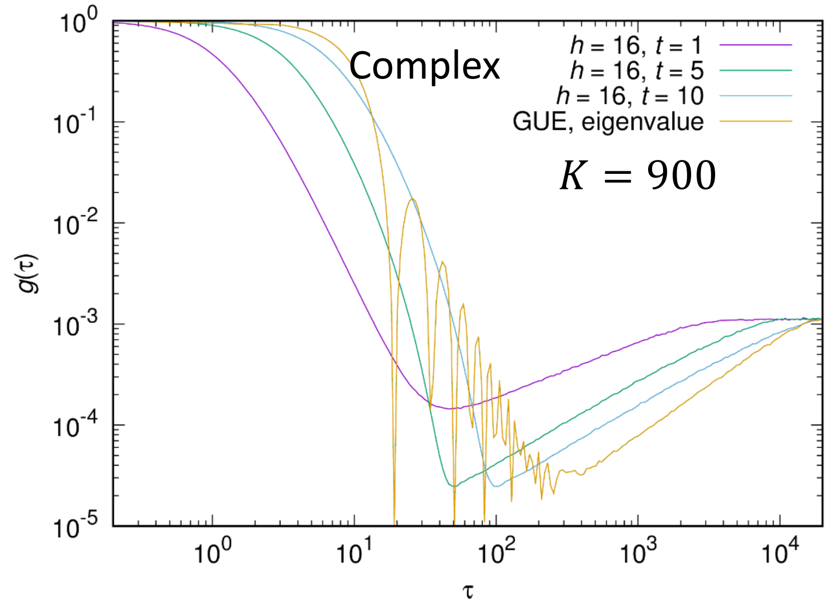
$$\left\{ \lambda_k(t) = \frac{\log a_k(t)}{t} \right\}_{k=1,2,\ldots,K}$$

1702.06935

$$T_{ij} = M^{(1)}_{ii'} M^{(2)}_{i'i''} \cdots M^{(t)}_{j'j} \qquad M^{(l)}_{ij} = 0 \quad \text{if } |i - j \bmod K| \geq h$$

Spectral form factor also approaches RMT result





Quantum case: how to define? Statistics? Physical interpretation?

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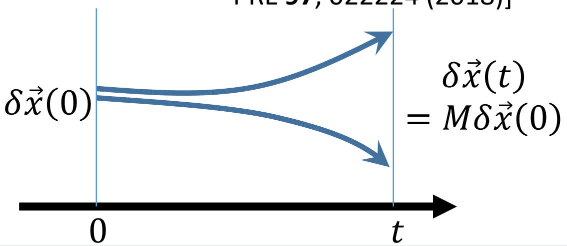
Quantum Lyapunov spectrum

Finite-time **classical Lyapunov spectrum**: obeys RMT statistics for chaos

[Hanada, Shimada, and MT: PRE **97**, 022224 (2018)]

Singular values of $M_{ij} = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)$ **at finite *t***: $\{s_k(t)\} = \{e^{\lambda_k t}\}$

$L = \{x_i(t), p_j(0)\}_{\text{PB}}^{\ 2} = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)^2 \to e^{2\lambda_{\text{L}} t}$ for large t



OTOC: $C_T(t) = \left\langle \left|[\hat{W}(t), \hat{V}(t=0)]\right|^2 \right\rangle = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle + \cdots$

Quantum Lyapunov spectrum: Define $\hat{M}_{ab}(t)$ as (anti)commutator of $\hat{O}_a(t)$ and $\hat{O}_b(0)$

$$\hat{L}_{ab}(t) = \left[\hat{M}(t)^\dagger \hat{M}(t)\right]_{ab} = \sum_{j=1}^{N} \hat{M}_{ja}(t)^\dagger \hat{M}_{jb}(t)$$

For $N \times N$ matrix $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$, obtain singular values $\{s_k(t)\}_{k=1}^N$.

The Lyapunov spectrum is defined as $\left\{\lambda_k(t) = \frac{\log s_k(t)}{2t}\right\}$.

Quantum Lyapunov spectrum for SYK model + modification

$$\hat{H} = \sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

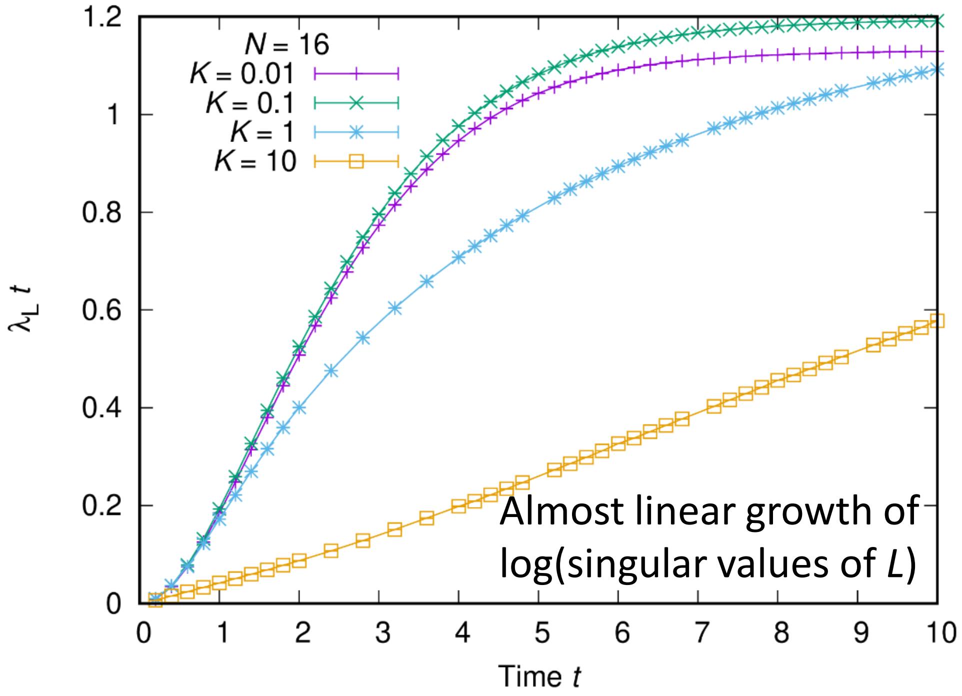
J_{abcd}: s. d. $= \frac{\sqrt{6}}{N^{3/2}}$

K_{ab}: s. d. $= \frac{K}{\sqrt{N}}$

- Define $\hat{L}_{ab}(t) = \sum_{j=1}^{N} \hat{M}_{ja}(t)\hat{M}_{jb}(t)$ for time-dependent anticommutator $\hat{M}_{ab}(t) = \{\hat{\chi}_a(t), \hat{\chi}_b(0)\}$.
- Obtain the singular values $\{a_k(t)\}_{k=1}^{K}$ of $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$
- Quantum Lyapunov spectrum: $\left\{\lambda_k(t) = \frac{\log a_k(t)}{2t}\right\}_{k=1,2,\ldots,K}$ (also dependent on state ϕ)

Other possibilities: see Rozenbaum-Ganeshan-Galitski, 1801.10591; Hallam-Morley-Green: 1806.05204

Growth of (largest Lyapunov exponent)*time



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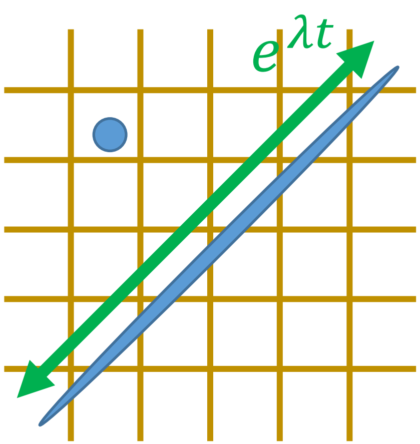
Kolmogorov-Sinai entropy vs entanglement entropy production

Coarse-grained entropy
= log(# of cells covering the region)
~ (sum of positive λ) t



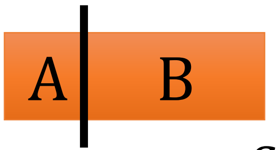
Kolmogorov-Sinai entropy h_{KS}
= (sum of positive λ)
= entropy production rate

Initial state with $S_{\mathrm{EE}} = 0$:

$$|\psi(t=0)\rangle = |000 \ldots 000\rangle$$

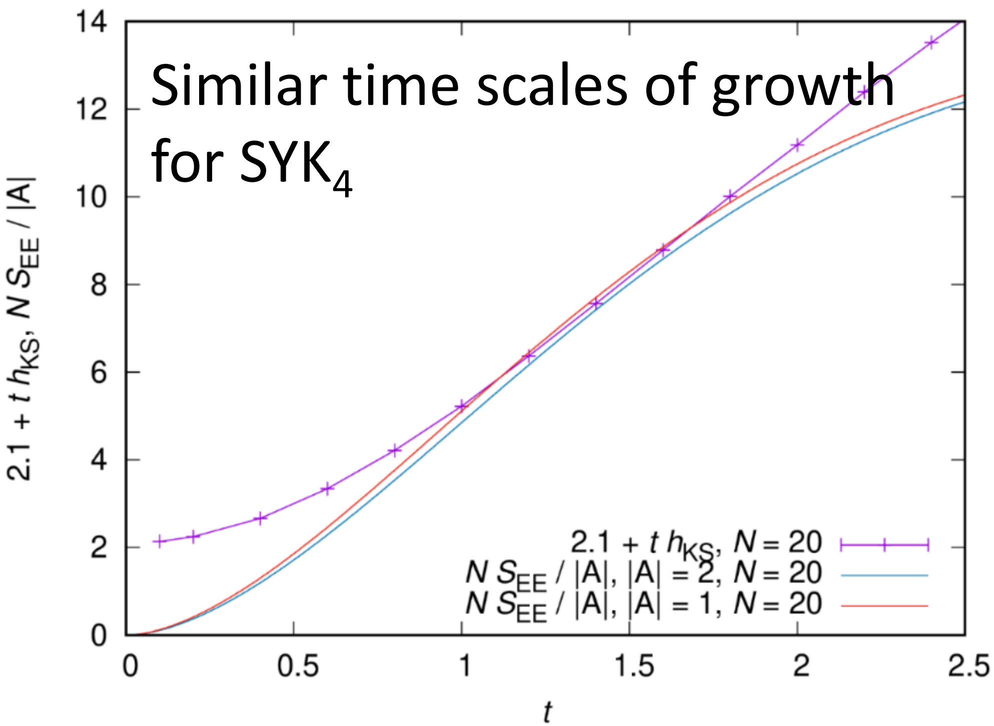
in the complex fermion basis

$$\hat{c}_j = \frac{(\chi_{2j-1} + \mathrm{i}\chi_{2j})}{\sqrt{2}}$$



$\rho_{\mathrm{A}}(t) = \mathrm{Tr}_{\mathrm{B}}\, \rho(t)$, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$

$$S_{\mathrm{EE}}(t) = -\mathrm{Tr}\, \rho_{\mathrm{A}}(t) \log\bigl(\rho_{\mathrm{A}}(t)\bigr)$$



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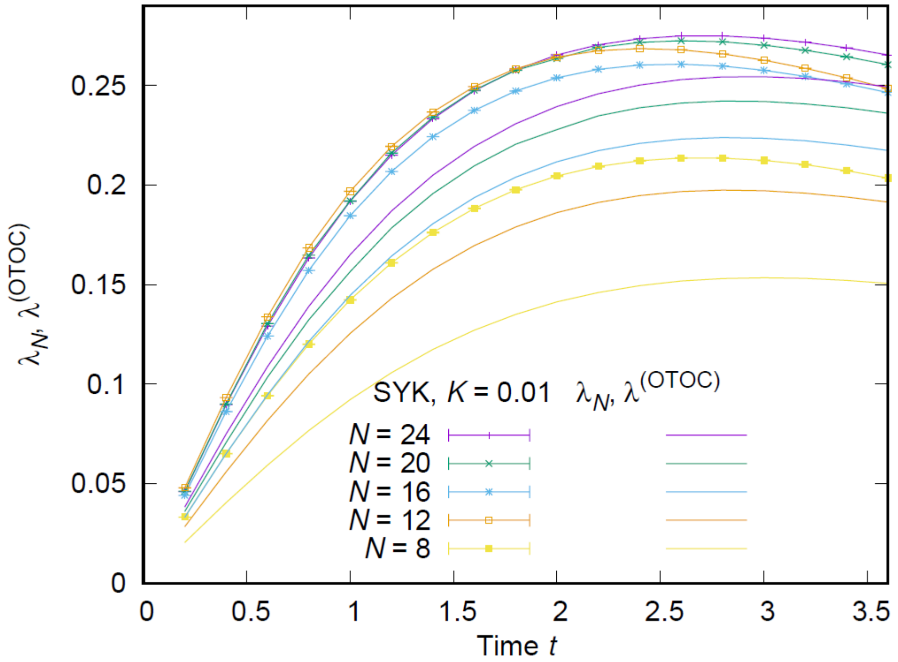
Fastest entropy production?

SYK$_4$ limit

- λ_N and $\lambda_{\rm OTOC} = \frac{1}{2t}\log\left(\frac{1}{N}\sum_{i=1}^{N} e^{2\lambda_i t}\right)$ approach each other; difference decreases as $1/N$
- Same for λ_N and λ_1:

  all exponent → single peak
- All saturate the MSS bound at strong coupling (low T) limit
- Growth rate of entanglement entropy $\sim h_{\rm KS}$ = sum of positive (all) λ_i



→ [conjecture] SYK model: not only the fastest scramblers, but also fastest entropy generators

Spectral statistics of quantum Lyapunov spectrum: SYK

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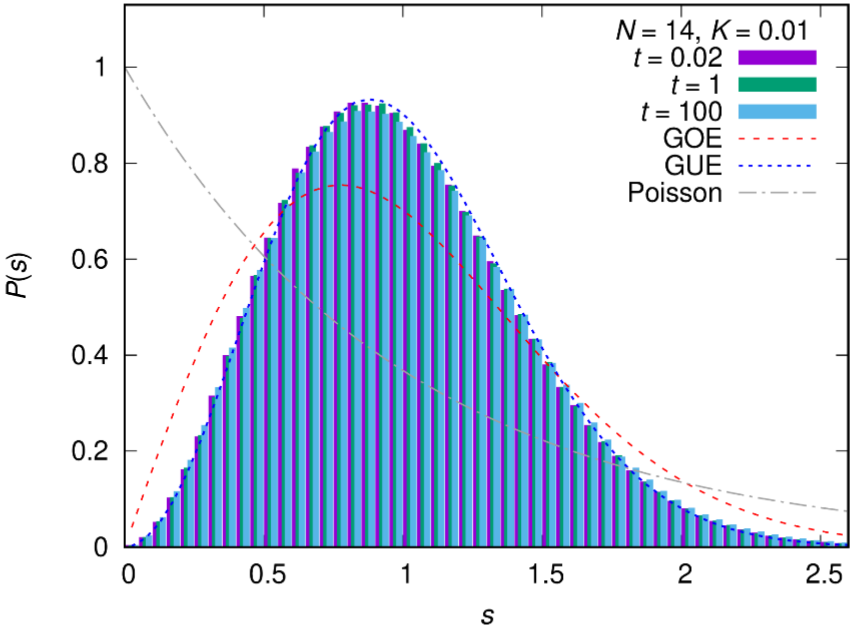
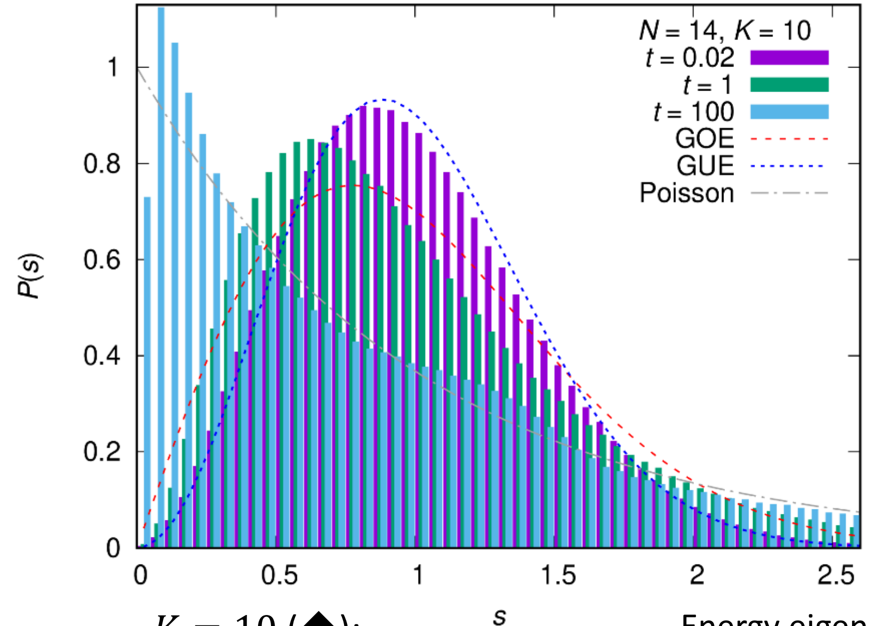
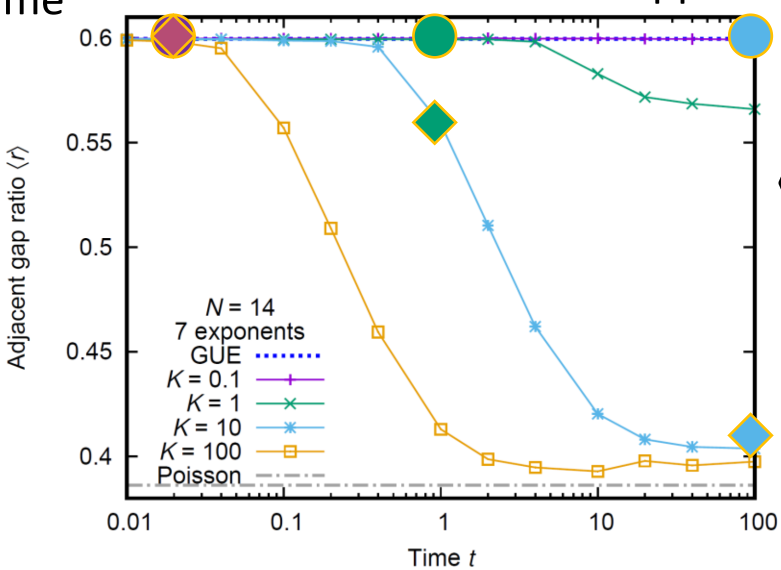




$K = 0.01$ (●):
Remains GUE for long time

$K = 10$ (◆):
Approaches Poisson

Energy eigenstates
$N/2$ larger exponents



$\langle r \rangle$: average of $\frac{\min(s_i, s_{i+1})}{\max(s_i, s_{i+1})}$

(fixed-i unfolding: unfold each gap $g_i = \lambda_{i+1} - \lambda_i$ using its average $\langle g_i \rangle_J$, $s_i = g_i / \langle g_i \rangle_J$)

The case of the random field XXZ model

$$\widehat{H} = \sum_{i}^{N} \widehat{\boldsymbol{S}}_i \cdot \widehat{\boldsymbol{S}}_{i+1} + \sum_{i}^{N} h_i \widehat{S_i^z}$$

h_i: uniform distribution $[-W, W]$

Many-body localization (MBL) transition at $W = W_c \sim 3.5$

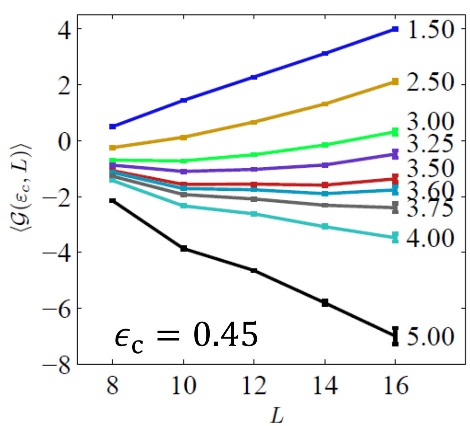
(though recently disputed; e.g. $W_c \geq 5$ proposed in E. V. H. Doggen et al., [1807.05051] using large systems with time-dependent variational principle & machine learning)

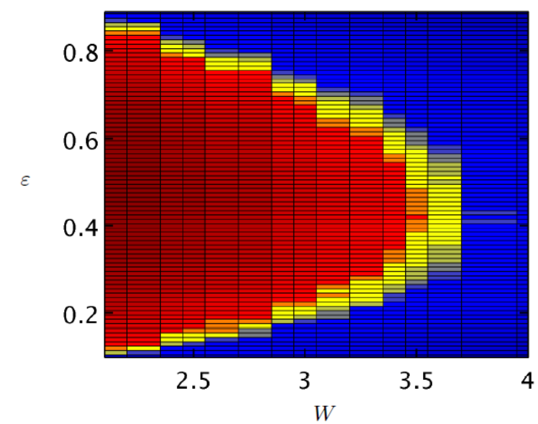
e.g. M. Serbyn, Z. Papic, and D. A. Abanin, Phys. Rev. X **5**, 041047 (2015) (arXiv:1507.01635)

Matrix element of local perturbation

$$\mathcal{G}(\varepsilon, L) = \ln \frac{|V_{n,n+1}|}{E'_{n+1} - E'_n}$$

Energy separation of neighboring energy eigenstates





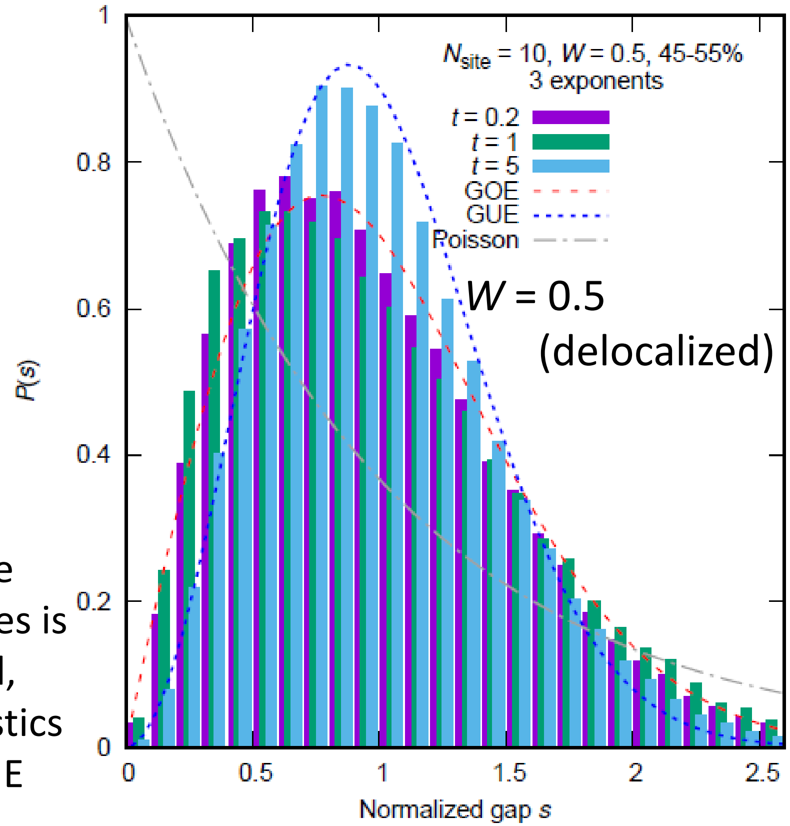
cf. MBL in short-range SYK [García-García and MT, Phys. Rev. B **99**, 054202 (2019)]; Localization of fermions on quasiperiodic lattice with attractive on-site interaction [Phys. Rev. A **82**, 043613 (2010)]

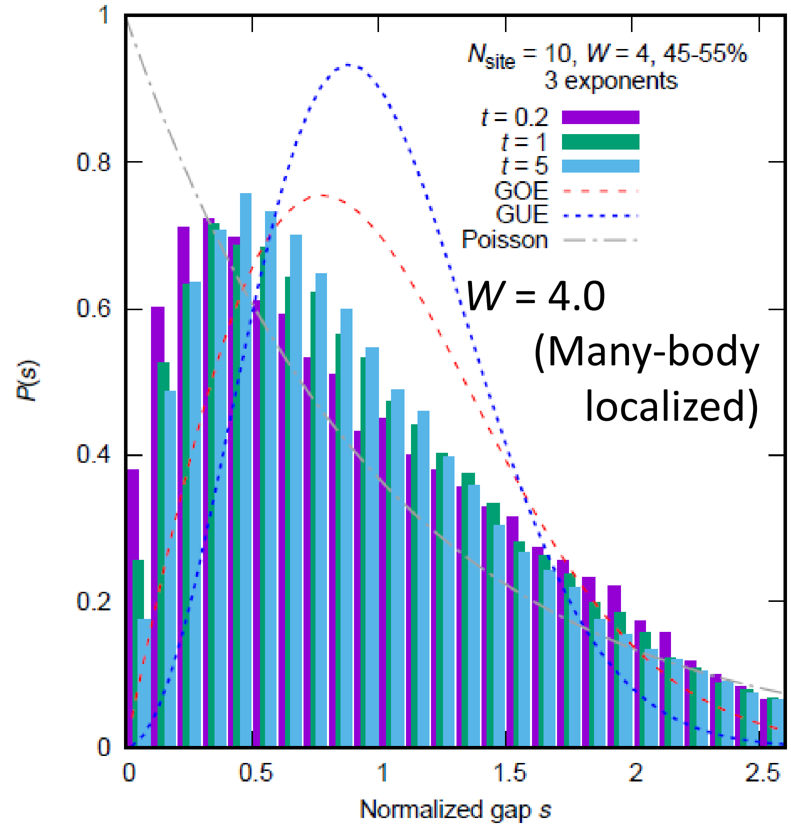
Spectral statistics of QLS for random field XXZ

$$\hat{H} = \sum_i^N \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1} + \sum_i^N h_i \widehat{S_i^z}$$

h_i: uniform distribution $[-W, W]$

$$\hat{M}_{ab}(t) = \left[\widehat{S_a^+}(t), \widehat{S_b^-}(0)\right]$$





- Exponential growth of the singular values is not observed, but the statistics approach GUE

Quantum Lyapunov spectrum distinguishes chaotic and non-chaotic phases

Singular value statistics of two-point time correlators

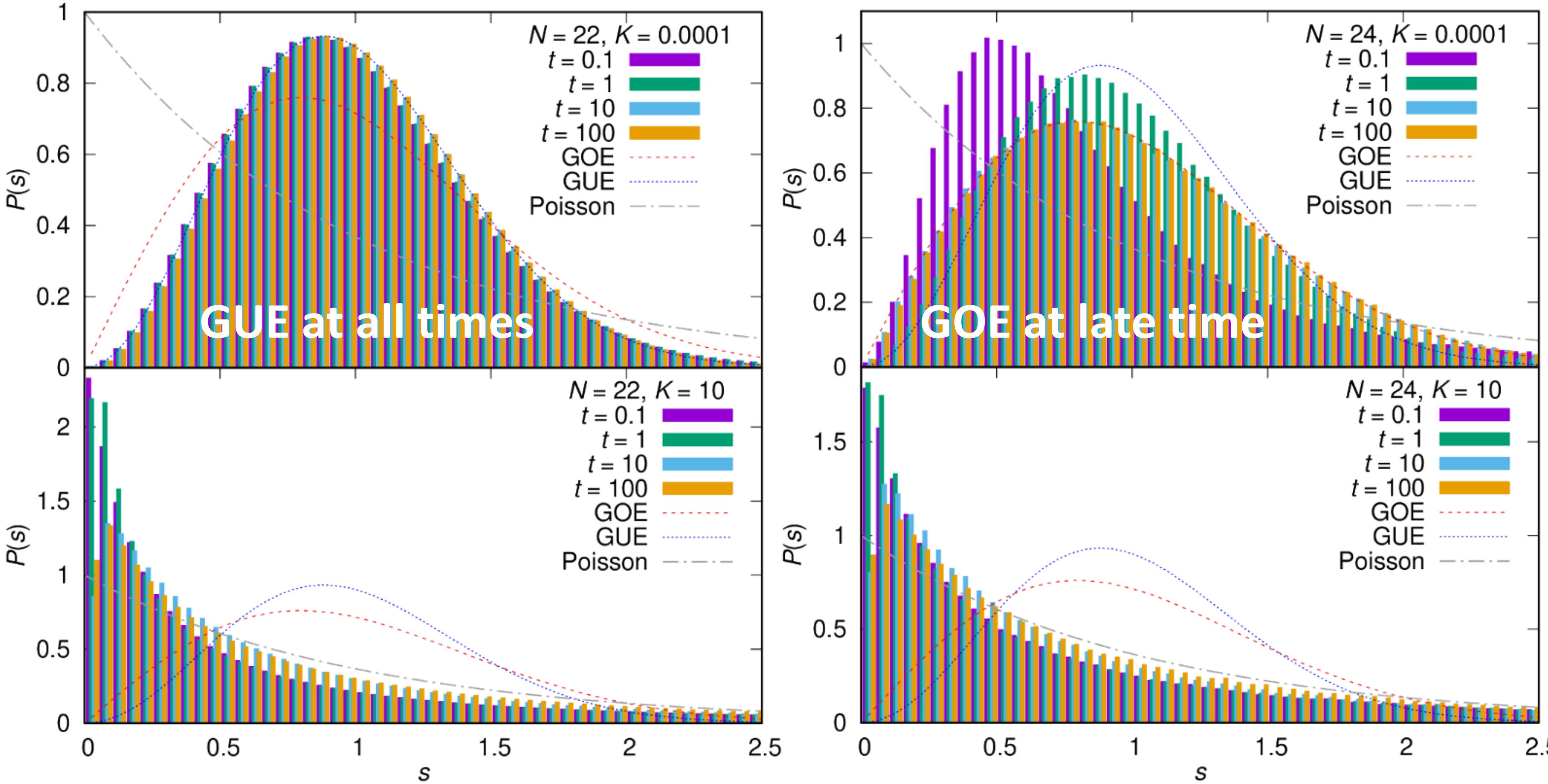
1902.11086

$$G_{ab}^{(\phi)}(t) = \langle \phi | \hat{\chi}_a(t) \hat{\chi}_b(0) | \phi \rangle \text{ as a matrix}$$

$$\lambda_j(t) = \log\left[\text{singular values of}\left(G_{ab}^{(\phi)}(t)\right)\right]$$

SYK$_4$

Strongly perturbed



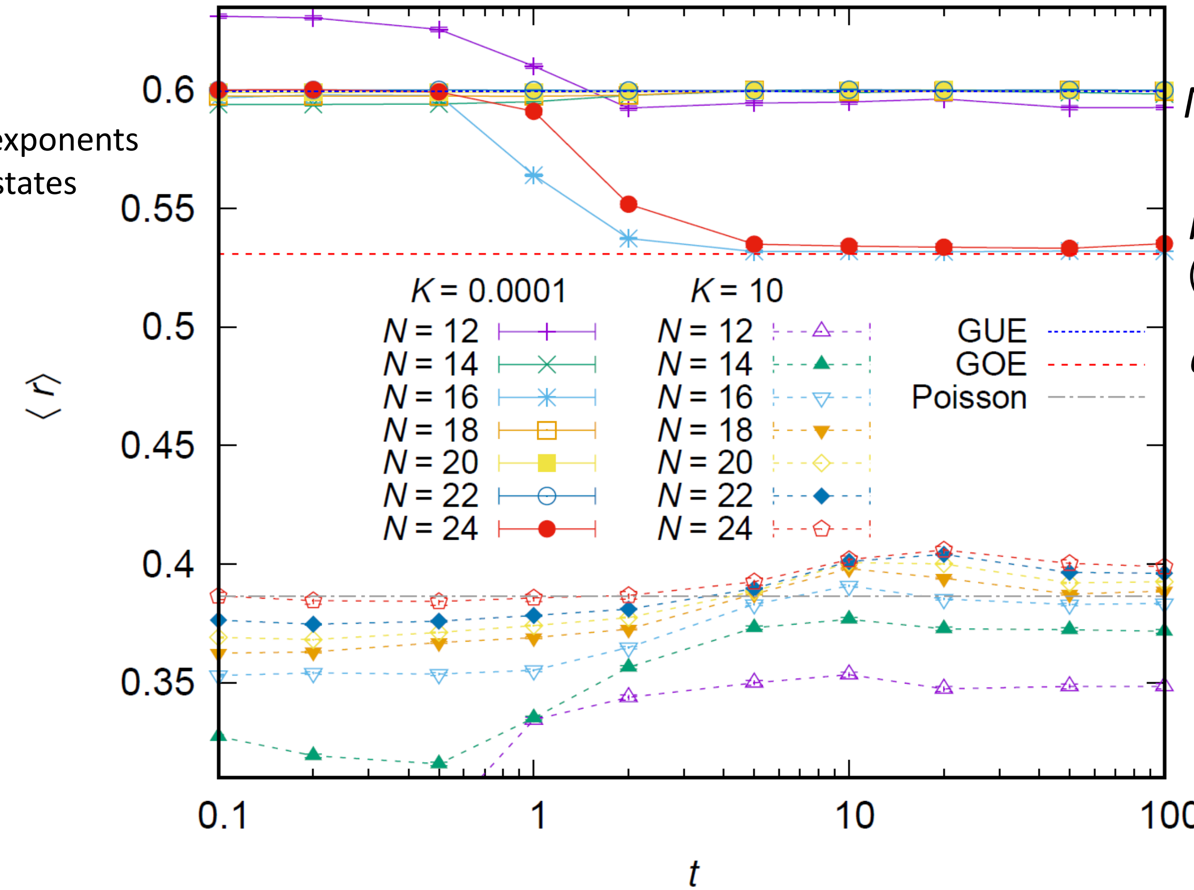
1902.11086

$\langle r \rangle$: average of the adjacent gap ratio $\frac{\min(\lambda_{i+1}-\lambda_i\,,\lambda_{i+2}-\lambda_{i+1})}{\max(\lambda_{i+1}-\lambda_i\,,\lambda_{i+2}-\lambda_{i+1})}$

Uncorrelated (Poisson): $2\log 2 - 1 \approx 0.386$

Correlated: larger (GOE: 0.5307, GUE: 0.5996 etc.) [Atas *et al.*, PRL 2013]

SYK, larger $N/2$ exponents
ϕ: energy eigenstates
fixed-i unfolded



N mod 8 = 2, 4, 6: GUE

N mod 8 = 0: GOE
(the matrix is symmetric)

$G_{ab}^{(\phi)} = \langle \phi | \hat{\chi}_a(t) \hat{\chi}_b(0) | \phi \rangle$

At late time, for two-point correlator singular values,
Random matrix behavior ⇔ chaotic

Summary

- Energy level correlation and OTOC reviewed for the Sachdev-Ye-Kitaev model, exponentially deep ramp followed by t^1 ramp; just ramp does not mean chaos
  - Cotler et al., JHEP05(2017)118; Gharibyan – Hanada – Shenker – Tezuka, JHEP07(2018)124; Lau – Ma – Murugan – Tezuka, Phys. Lett. B in press
- SYK model with an additional hopping term: chaos → integrable transition
  - García-García – Loureiro – Romero-Bermudez – Tezuka, PRL 120, 241603 (2018)
- Finite-time Lyapunov spectrum
  - Hanada-Shimada-Tezuka, Phys. Rev. E 97, 022224 (2018)
- Quantum Lyapunov spectrum
  - Gharibyan-Hanada-Swingle-Tezuka, JHEP04(2019)082
- Singular value spectrum of two-point time correlators
  - Gharibyan-Hanada-Swingle-Tezuka, arXiv:1902.11086

Спасибо за ваше внимание. Je vous remercie pour votre aimable attention.

Example (0a): Logistic map

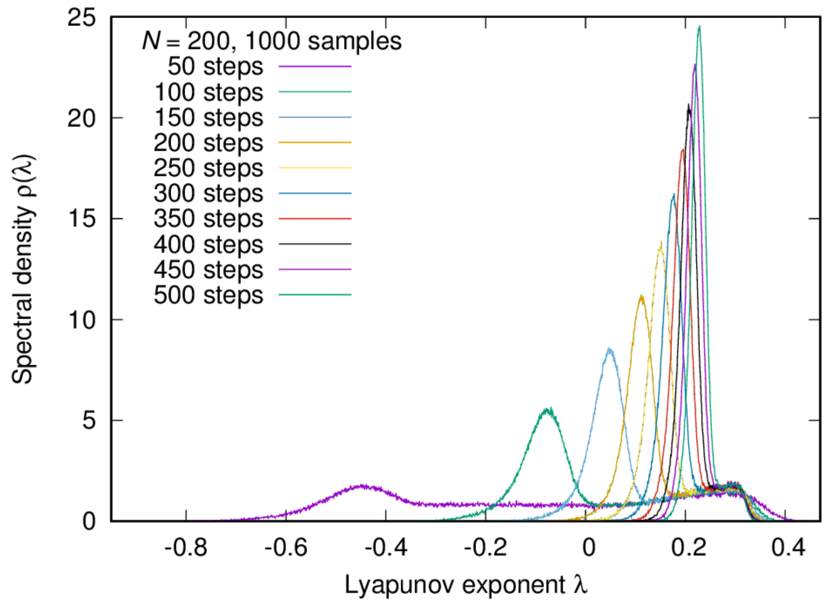
$$x_i(t+1) = 1 - a x_i(t)^2$$

+ coupling

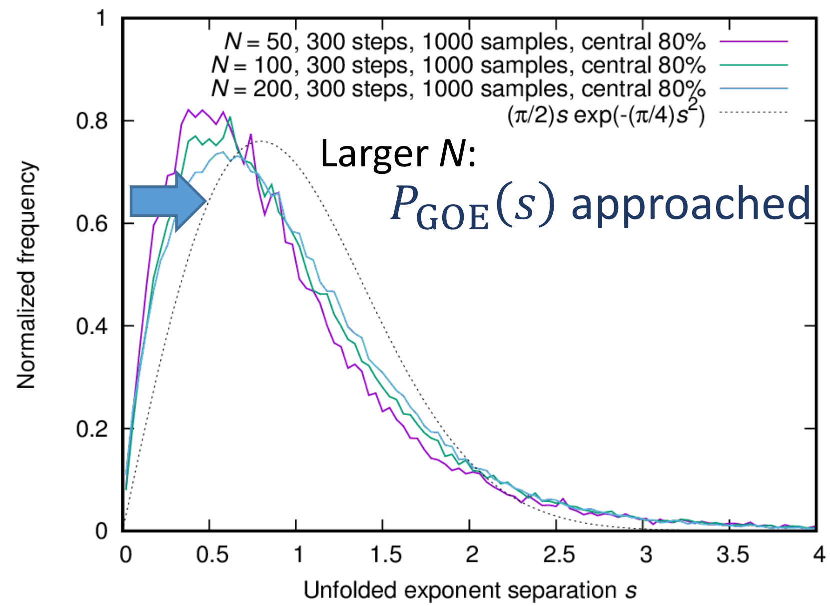
$$x_i(t+1) = (1-e)(1 - a x_i(t)^2) + \frac{e}{2}\left((1 - a x_{i-1}(t)^2) + (1 - a x_{i+1}(t)^2)\right)$$

$a = 1.83$ (chaotic)   coupling: $e = 0.3$

Lyapunov spectrum $\rho(\lambda)$



Separation distribution $P(s)$



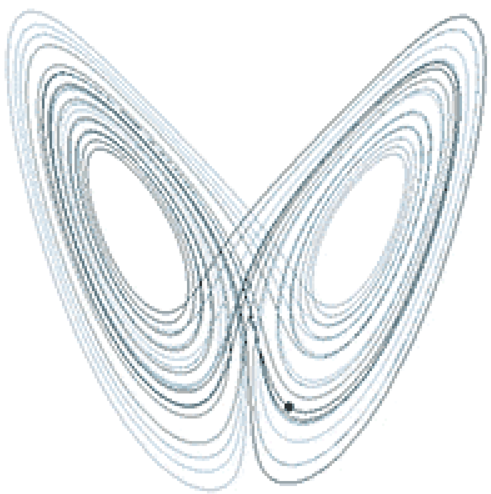
Example (0b): Lorenz attractor

$$\dot{X}_i = -pX_i + pY_i$$
$$\dot{Y}_i = -X_iZ_i + rX_i - Y_i$$
$$\dot{Z}_i = X_iY_i - bZ_i$$
$$(p, r, b) = (10, 28, 8/3)$$

E. N. Lorenz (1963)

+ coupling

$$\dot{X}_i = -pX_i + pY_i + c\left(X_i - \frac{X_{i-1} + X_{i+1}}{2}\right)$$
$$\dot{Y}_i = -X_iZ_i + rX_i - Y_i + c\left(Y_i - \frac{Y_{i-1} + Y_{i+1}}{2}\right)$$
$$\dot{Z}_i = X_iY_i - bZ_i + c\left(Z_i - \frac{Z_{i-1} + Z_{i+1}}{2}\right)$$



[Dan Quinn, Wikimedia Commons]

coupling $c = 0.5$