引力全息与量子物质研讨会

Workshop on Holography and Quantum Matter

Institute of Theoretical Physics,
Chinese Academy of Sciences, Beijing

GRADUATE SCHOOL OF / FACULTY OF SCIENCE KYOTO UNIVERSITY

The Sachdev-Ye-Kitaev model as a maximally chaotic lattice model: study towards experimental realization and new characterizations of chaos

26 August 2019

Masaki **TEZUKA** （**手塚**真樹）

(Kyoto University)

Contents

$$\hat{H}_{\mathrm{SYK}} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

black hole

holography

non-gravitational quantum system

cold atom realization

Danshita, Hanada and MT, PTEP 2017 [1606.02454]

maximally chaotic (chaos bound)

maximally chaotic (chaos bound)

$\lambda_{\mathrm{Lyapunov}}$

numerical analysis; relation to other scrambling systems

Cotler et al., JHEP 2017 [1611.04650]
Gharibyan et al., JHEP 2018 [1803.08050]

random matrix behavior of finite-time Lyapunov spectrum

[Classical] Hanada, Shimada and MT, PRE 2018 [1702.02197]

[Quantum] Gharibyan, Hanada, Swingle and MT, JHEP 2019 [1809.01671]

modifications to study chaos / integrable transition & many-body localization

García-García et al., PRL 2018 [1707.02197]

García-García and MT, PRB 2019 [1801.03204]

random matrix behavior of two-point correlators

Lau, Ma, Murugan, and MT, Phys. Lett. B 2019 [1812.04770]

Gharibyan, Hanada, Swingle and MT, 1902.11086

Collaborators (in SYK-related papers) and references

- Jordan Saul Cotler[a], Guy Gur-Ari[a] (→Google), **Masanori Hanada** (YITP→Boulder→Southampton)
- Joseph Polchinski[b], Phil Saad[a], Stephen H. Shenker[a], Douglas Stanford[a], Alexandre Streicher[b]
- Ippei Danshita (YITP→Kindai), Hidehiko Shimada (OIST), Hrant Gharibyan[a], Brian Swingle (Maryland)
- Antonio M. García-García (SJTU), Bruno Loureiro (Cambridge), Aurelio Romero-Bermúdez (Leiden)
- Pak Hang Chris Lau (MIT→NTHU), **Chen-Te Ma** (SCNU & Cape Town), Jeff Murugan (Cape Town & KITP)

[a]Stanford [b]UCSB

Danshita, Hanada, and MT, PTEP 2017, 083I01 (arXiv:**1606.02454**)

Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher, and MT, JHEP 1705, 118 (2017) (arXiv:1611.04650)

Hanada, Shimada, and MT, Phys. Rev. E **97**, 022224 (2018) (arXiv:1702.06935)

García-García, Loureiro, Romero-Bermudez, and MT, PRL **120**, 241603 (2018) (arXiv:**1707.02197**)

García-García and MT, Phys. Rev. B **99**, 054202 (2019) (arXiv:**1801.03204**)

Gharibyan, Hanada, Shenker, and MT, JHEP 1807, 124 (2018) (arXiv:1803.08050)

Gharibyan, Hanada, Swingle, and MT, JHEP 1904, 082 (2019) (arXiv:**1809.01671**), submitted (arXiv:**1902.11086**)

Lau, Ma, Murugan, and MT, Phys. Lett. B **795**, 230 (10 August 2019) (arXiv:1812.04770)

The Sachdev-Ye-Kitaev model

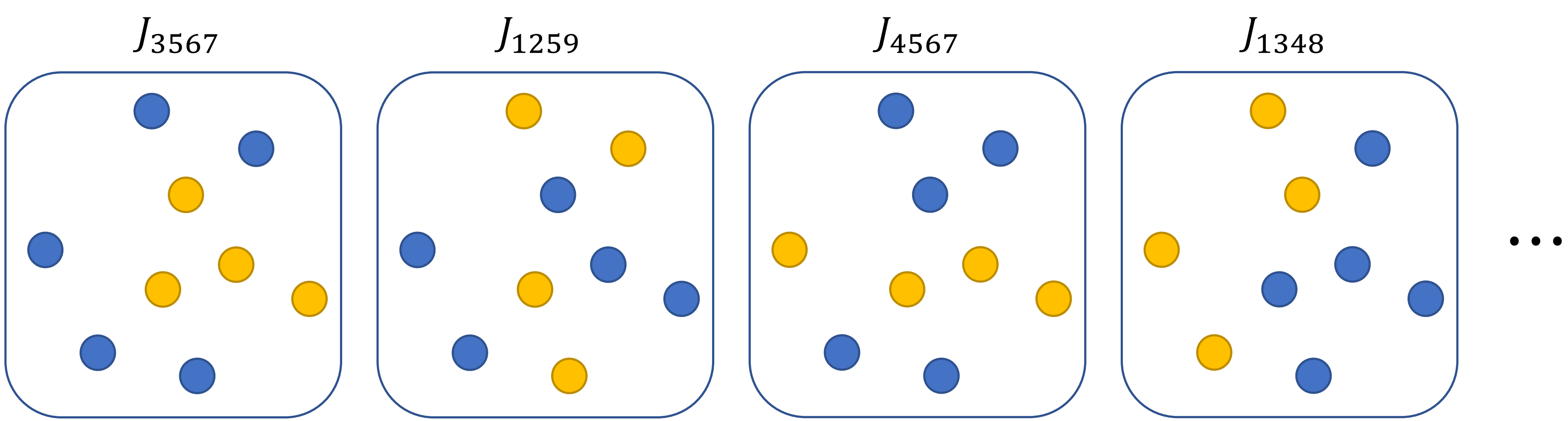
$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

cf. Sachdev-Ye model (1993)

[A. Kitaev, talks at KITP (2015)]

$\hat{\chi}_{a=1,2,\ldots,N}$: N Majorana fermions ($\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$)

J_{abcd} : independent Gaussian random couplings ($\langle {J_{abcd}}^2 \rangle = J^2 = 1$)



Two versions of the SYK model

N Majorana- or Dirac- fermions randomly coupled to each other

[Majorana version]

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a<b<c<d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

[A. Kitaev: talks at KITP (Feb 12, Apr 7 and May 27, 2015)]

[Dirac version]

$$\hat{H} = \frac{1}{(2N)^{3/2}} \sum_{ij;kl} J_{ij;kl} \hat{c}_i{}^{\dagger} \hat{c}_j{}^{\dagger} \hat{c}_k \hat{c}_l$$

[A. Kitaev's talk]
[S. Sachdev: PRX **5**, 041025 (2015)]

Studied as "Two-body random ensemble" since 1970s

(The first paper by A. Kitaev on the SYK model:
Alexei Kitaev and S. Josephine Suh, arXiv:1711.08467 (JHEP**05**(2018)183);
First papers by J. Ye on the SYK model: arXiv:1809.06667 and arXiv:1809.07577)

Why solvable in the $N \gg 1$ limit?

(after sample average $\langle \cdots \rangle_{\{J\}}$)

Free two-point function $G_0(t)\delta_{ij} = -\langle \mathrm{T}\psi_i(t)\psi_j(0)\rangle = -\frac{1}{2}\mathrm{sgn}(t)\delta_{ij}$

Perturbation expansion by interaction term

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

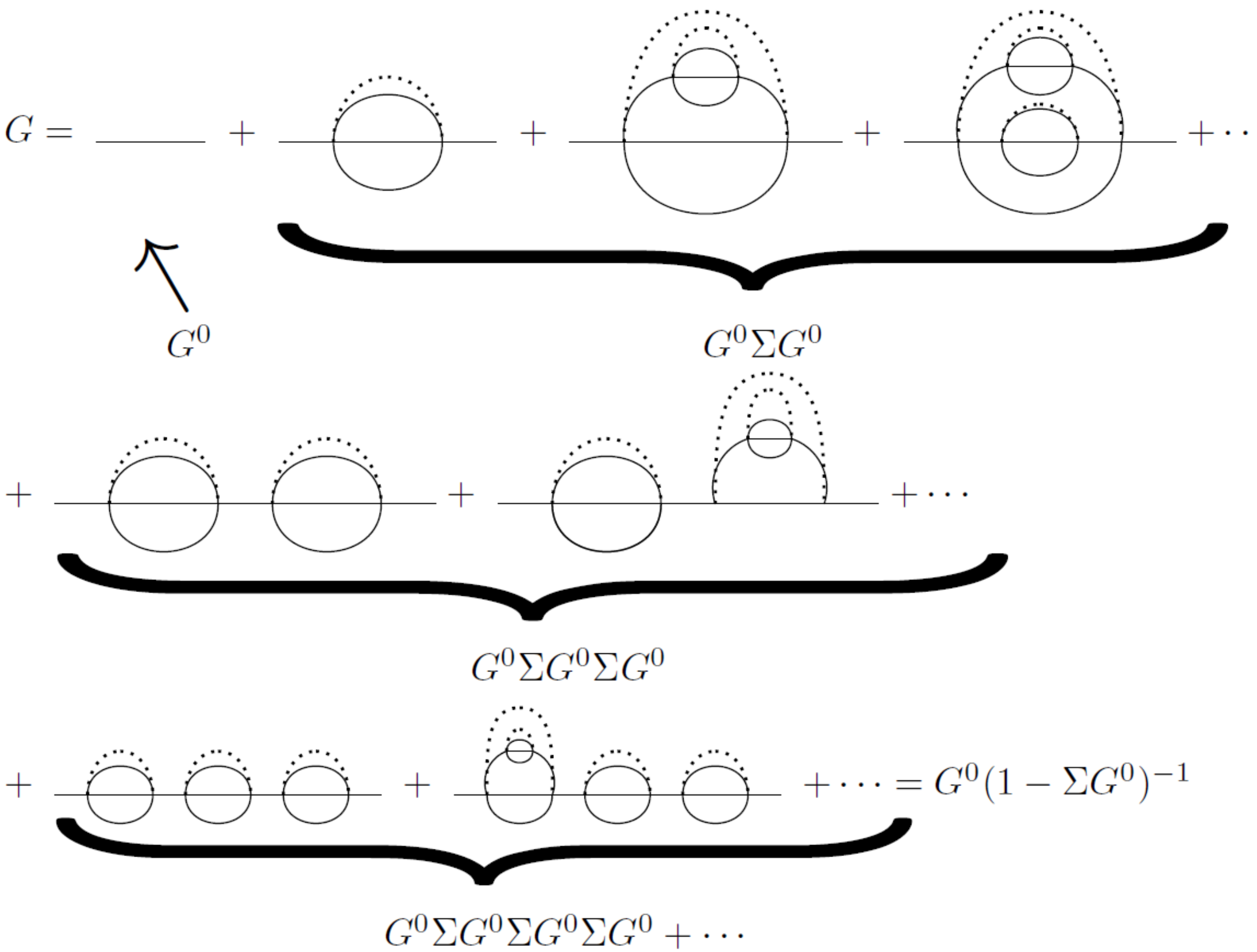
$\langle {J_{abcd}}^2 \rangle_{\{J\}} = J^2$, independent Gaussian distribution

$\langle J_{abcd}J_{abce}\rangle_{\{J\}} = 0$ if $d \neq e$ ➔ Most diagrams average to zero

“Melon-type” diagrams dominate in large *N*

“Melon” diagrams dominate in the $N \gg 1$ limit

Dotted line connects same couplings




第73巻第8号(通巻835号)
PRINT ISSN 0029-0181
ONLINE ISSN 2423-8872
昭和30年6月13日 第3種郵便物承認
平成30年8月5日発行 毎月5日発行

•強は異なり
•シグマハイパー核はどこまでわかったか?
•蒸発と乾燥の物理学
•銅酸化物高温超伝導体における
擬ギャップと磁場誘起電荷密度波

NO. 8
2018 | VOL. 73

B U T S U R I
日本物理学会誌

最近の研究から

Sachdev–Ye–Kitaev 模型，ブラックホール，冷却気体系

段 下 一 平 (近畿大学理工学部 danshita@phys.kindai.ac.jp)
手 塚 真 樹 (京都大学大学院理学研究科 tezuka@scphys.kyoto-u.ac.jp)
花 田 政 範 (京都大学基礎物理学研究所，白眉センター hanada@yukawa.kyoto-u.ac.jp)

超弦理論は重力の量子論の有力な候補として長年研究され続けている．究極の目標は自然界のあらゆる相互作用を統一する万物の理論の構築だが，副産物として数学や物理学の様々な分野との繋がりが見出されてきた．特に，この2，3年で物性理論，量子情報理論と超弦理論の意外な関係が明らかになってきた．

超弦理論を非摂動的にどう定義したらよいかというのは長年の問題だが，重力を含まないある種の量子場の理論が超弦理論，あるいはより一般的な量子重力理論の定義になっているのではないかというホログラフィー原理という考えがここ20年ほど有力視されている．対応が最もよく理解されているのは極大超対称ゲージ理論と呼ばれる一見特殊な理論の場合だが，最近，SYK模型という物性分野から出てきた理論が重力理論の少なくともある種の特徴をとらえていることがわかってきた．

SYK模型は，N個のフェルミオンが非局所的にランダムに相互作用している模型である．元々は1990年代初頭にサチデフ（Sachdev）と叶（Ye）が銅酸化物高温超伝導体の関連物質の実験に関係して非フェルミ液体状態を記述するために提案したSY模型というものがあったが，SYK模型はこれを簡単化してキタエフ（Kitaev）が2015年に提案した模型である．サチデフはもともと物性理論への応用という立場からホログラフィー原理に興味を持っていたようだが，途中から，SY模型を使って量子重力理論を定義するという方向性も追求し始めた．2015年にキタエフがSYK模型が「カオスの上限」を実現することを示し，量子重力の観点からの研究に火が付いた．

現在では，SYK模型と対応する重力理論が何かはまだわかっていないものの，量子重力や量子カオスの研究の舞台として積極的に研究され，また，関連する模型も多々提案されている．著者らは，光格子中の冷却気体を用いてSYK模型を実現する方法を提案した．この方法では，深い光格子の1サイトに複数のフェルミ原子を捕獲し，光会合レーザーにより，任意の2準位から分子状態への遷移を可能とする．形成された分子が別の2準位の原子へと速やかに光解離する状況で，分子の内部自由度を活用することにより，必要な相互作用のランダムさを実現できるという提案である．

SYK模型や超対称ゲージ理論のような量子重力理論の定義となると目されている理論を実験的に実現することができたとすると，量子重力系の様々な性質，たとえばブラックホールの生成や蒸発などを実験的に調べることができると期待される．そのような意味で，物性理論や冷却気体実験の専門家が，量子重力の研究に貢献できる可能性が拓かれつつある．

Figure from [I. Danshita, M. Tezuka, and M. Hanada: Butsuri **73**(8), 569 (2018)]

Feynman diagrams

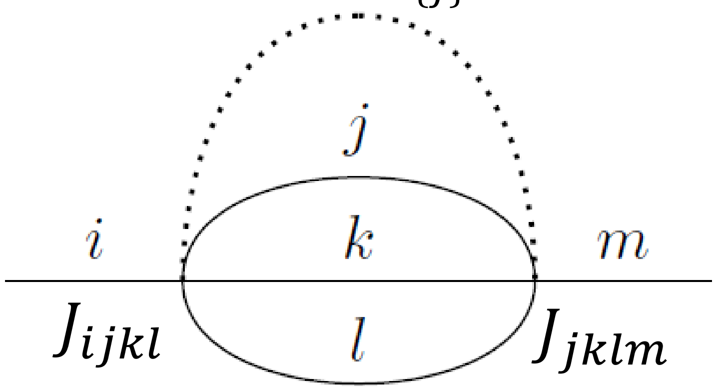
[J. Polchinski and V. Rosenhaus, JHEP 1604 (2016) 001]
[J. Maldacena and D. Stanford, PRD **94**, 106002 (2016)]

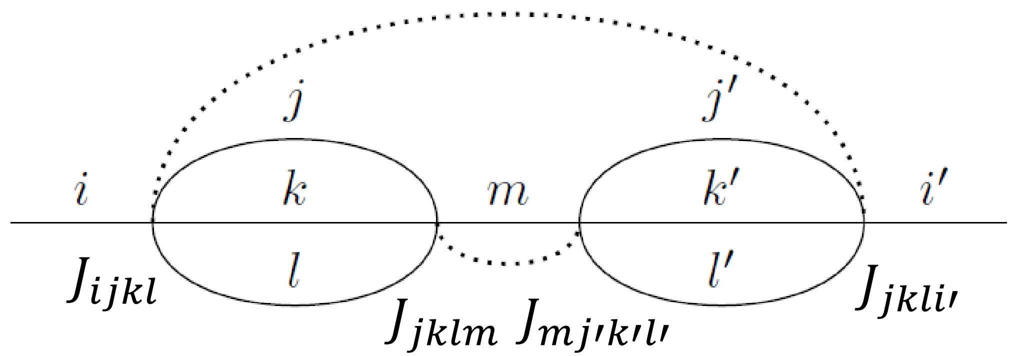
$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a<b<c<d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

$$\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$$

$$\langle J_{abcd}{}^2 \rangle = J^2 = 1$$

Sample average $\langle \cdots \rangle_{\{J\}}$





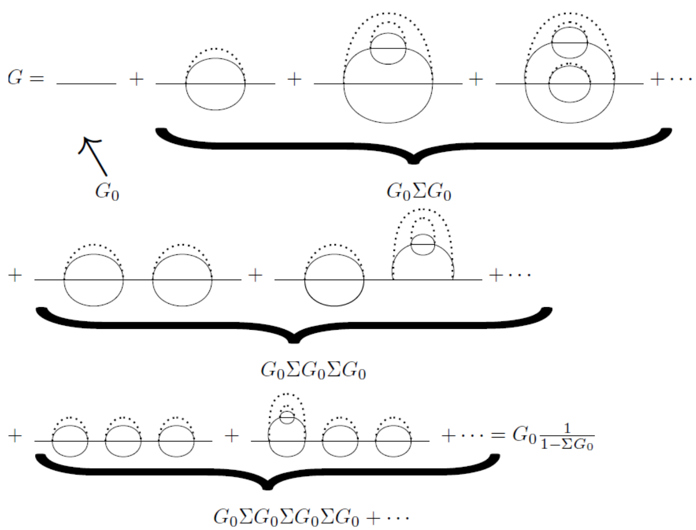
$$\sum_{jkl} \langle J_{ijkl} J_{jklm} \rangle_{\{J\}} = \frac{N^3}{3!} \delta_{im}$$

$$\sum_{m \neq i} \sum_{jklj'k'l'} \langle J_{ijkl} J_{jklm} J_{mj'k'l'} J_{j'k'l'i'} \rangle_{\{J\}} \propto N^4 \delta_{ii'}$$

$\Rightarrow$ $O(N^0)$ contribution

$\Rightarrow$ $O(N^{-2})$ contribution

Reparametrization symmetry



$$G(1-\Sigma G_0) = G_0$$

$$G^{-1} = G_0{}^{-1} - \Sigma$$

$$G(i\omega)^{-1} = i\omega - \Sigma(i\omega) \qquad \Sigma = J^2 G^3$$

Low energy ($\omega, T \ll J$): ignore $i\omega$ and we have

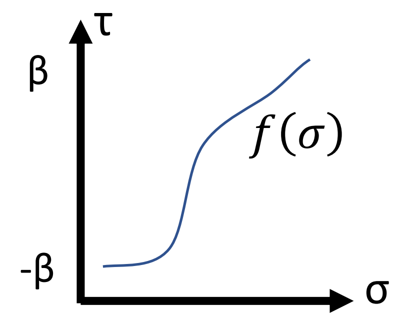
$$\int dt G(t_1, t)\Sigma(t, t_2) = -\delta(t_1, t_2)$$

Invariant under imaginary time reparametrization

$$\tau = f(\sigma),$$

$$G(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-1/4} \frac{g(\sigma_1)}{g(\sigma_2)} G(\sigma_1, \sigma_2),$$

$$\tilde{\Sigma}(\tau_1, \tau_2) = [f'(\sigma_1) f'(\sigma_2)]^{-3/4} \frac{g(\sigma_1)}{g(\sigma_2)} \tilde{\Sigma}(\sigma_1, \sigma_2),$$



Large-N saddle point solution

$$\int dt G(t_1, t) \Sigma(t, t_2) = -\delta(t_1, t_2)$$

(Derived in replica formalism; assume replica symmetry)

$$G_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-1/2} \quad , \quad \Sigma_s(\tau_1 - \tau_2) \sim (\tau_1 - \tau_2)^{-3/2}$$

Not invariant under arbitrary reparametrization,
but invariant under

$$f(\tau) = \frac{a\tau + b}{c\tau + d} \quad , \quad ad - bc = 1.$$

Symmetry broken to $SL(2, R)$.
cf. isometry group of AdS_2
[see e.g. A. Strominger, hep-th/9809027]

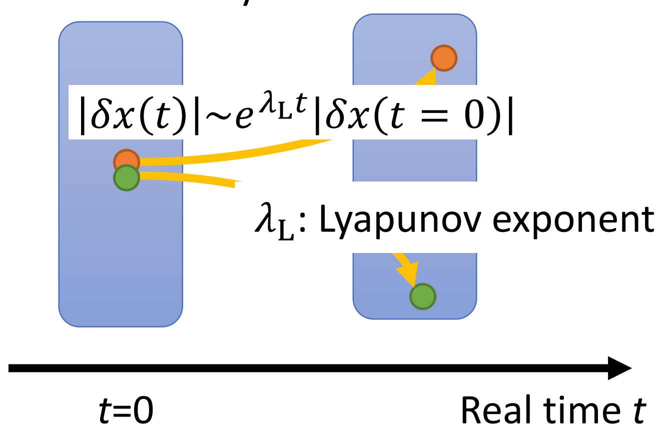
[J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)]
Study of the Goldstone modes: *e.g.* [D. Bagrets, A. Altland, and A. Kamenev, Nucl. Phys. B **911**, 191 (2016)]

SYK: Nearly CFT_1 "$NCFT_1$"
emergent conformal gauge invariance
[Sachdev, PRX **5**, 041025 (2015)]

Definition of Lyapunov exponent using out-of-time-order correlators

$$F(t) = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle \qquad W(t) = \mathrm{e}^{iHt}W\mathrm{e}^{-iHt}$$

Classical:
Infinitesimally different initial states



$$\{x(t), p(0)\}_{\mathrm{PB}}{}^2 = \left(\frac{\partial x(t)}{\partial x(0)}\right)^2 \to e^{2\lambda_{\mathrm{L}} t}$$

Consider operators V and W,

$$C(t) = \langle |[W(t), V(t=0)]|^2\rangle$$
$$= 2\big(1 - \mathrm{Re}\,F(t)\big)$$

quantifies strength of quantum scrambling

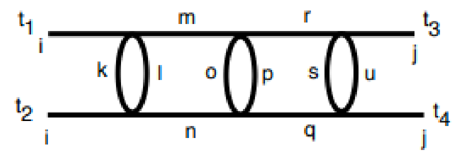
"Black holes are fastest quantum scramblers"

[P. Hayden and J. Preskill 2007] [Y. Sekino and L. Susskind 2008] [Shenker and Stanford 2014]

Chaos bound $\lambda_{\mathrm{L}} = 2\pi k_{\mathrm{B}} T/\hbar$

[J. Maldacena, S. H. Shenker, and D. Stanford, JHEP08(2016)106]

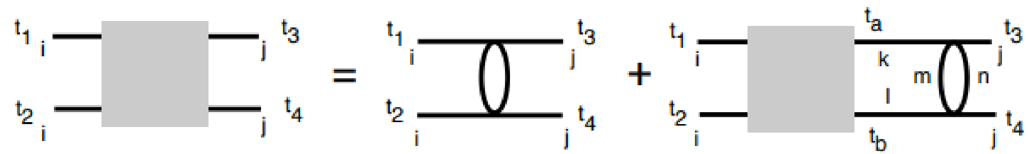
Out-of-time-ordered correlators (OTOCs)



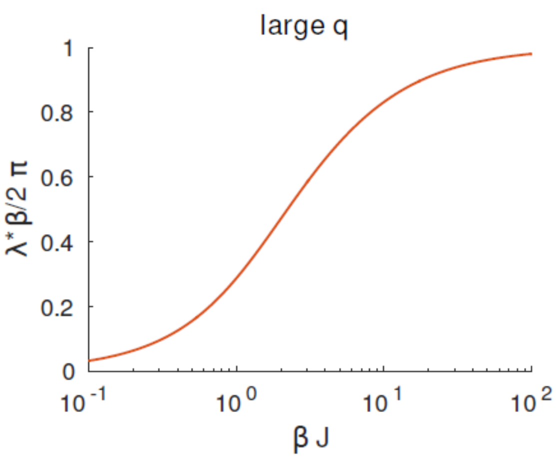
$\langle \hat{\chi}_i(t_1)\hat{\chi}_i(t_2)\hat{\chi}_j(t_3)\hat{\chi}_j(t_4)\rangle$

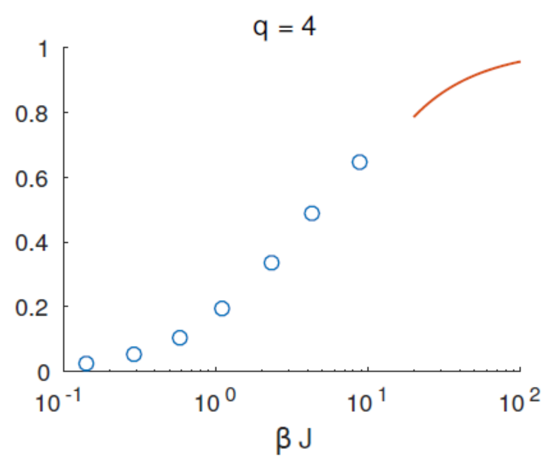
(a)

Regularized OTOC can be calculated for large-*N* SYK model, satisfies the chaos bound $\lambda_{\mathrm{L}} = 2\pi k_{\mathrm{B}}T/\hbar$ at low *T* limit



$$\Gamma(t_1,t_2,t_3,t_4) = \Gamma_0(t_1,t_2,t_3,t_4) + \int dt_a dt_b\, \Gamma(t_1,t_2,t_a,t_b)K(t_a,t_b,t_3,t_4)$$

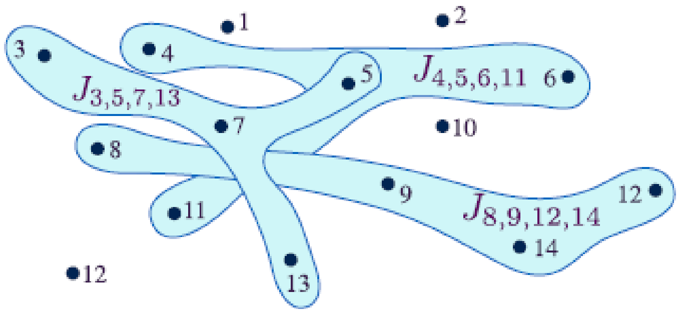




[Kitaev's talks]
[J. Polchinski and V. Rosenhaus, JHEP 1604 (2016) 001]
[J. Maldacena and D. Stanford, Phys. Rev. D **94**, 106002 (2016)]

Holographic connection to gravity

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$



$$\mathcal{Q} = \frac{1}{N} \sum_i \left\langle c_i^\dagger c_i \right\rangle .$$

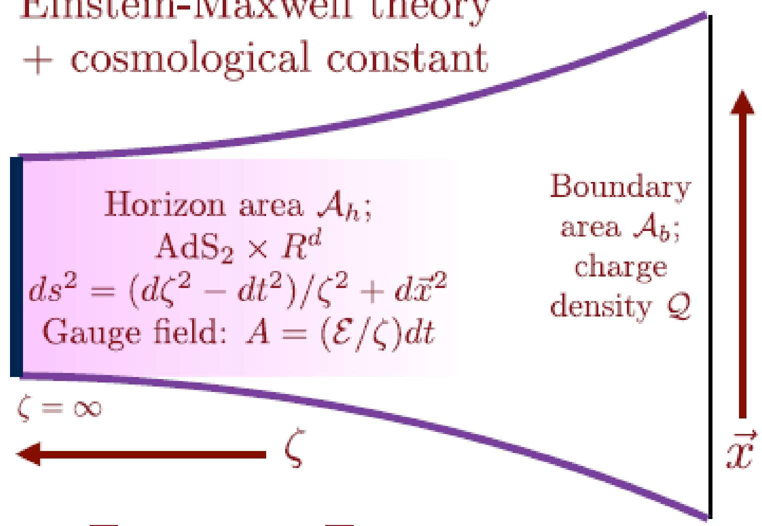
$$-\left\langle c_i(\tau) c_i^\dagger(0) \right\rangle \sim \begin{cases} -\tau^{-1/2} & , \tau > 0 \\ e^{-2\pi\mathcal{E}}\, |\tau|^{-1/2} & , \tau < 0. \end{cases}$$

Known "equation of state" determines $\mathcal{E}$ as a function of $\mathcal{Q}$

Microscopic zero temperature entropy density $\mathcal{S}$ obeys

$$\frac{\partial \mathcal{S}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$$

Einstein-Maxwell theory + cosmological constant



$$\mathcal{L} = \overline{\psi}\Gamma^\alpha D_\alpha \psi + m\overline{\psi}\psi$$

$$-\left\langle \psi(\tau)\overline{\psi}(0) \right\rangle \sim \begin{cases} -\tau^{-1/2} & , \tau > 0 \\ e^{-2\pi\mathcal{E}}\, |\tau|^{-1/2} & , \tau < 0. \end{cases}$$

"Equation of state" relating $\mathcal{E}$ and $\mathcal{Q}$ depends upon the geometry of spacetime far from the AdS_2

Black hole thermodynamics (classical general relativity) yields

$$\frac{\partial \mathcal{S}_{\rm BH}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$$

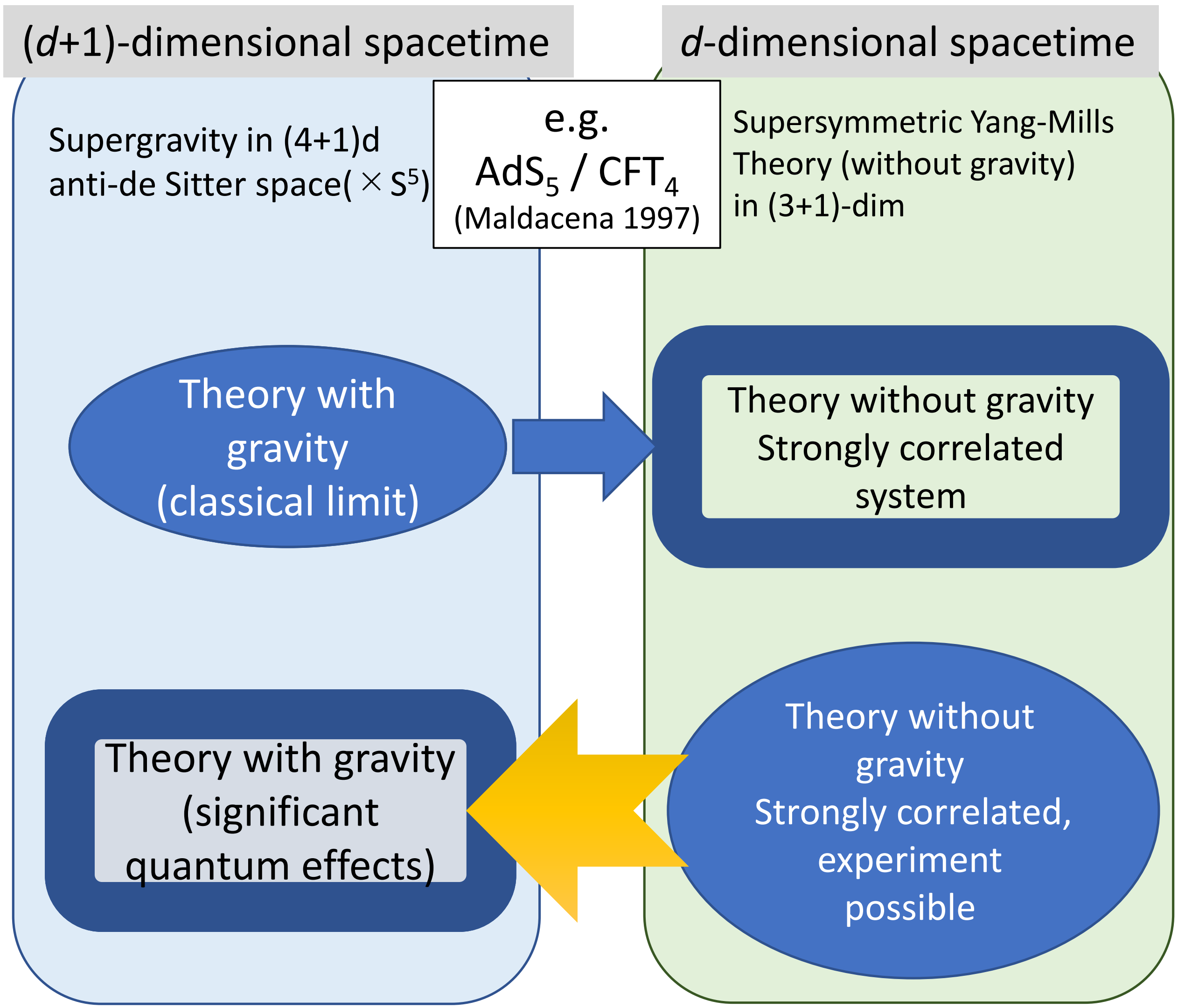
[S. Sachdev, Phys. Rev. X **5**, 041025 (2015)]

Contents

- The Sachdev-Ye-Kitaev model
  - ✓ Large-N solvability: conformal symmetry and maximal chaos
  - Experimental proposal 1606.02454 (and realization)
  - Deformation and suppression of maximal chaos 1707.02197
- Characterization of chaos in random systems 1702.06935
  - Quantum Lyapunov spectrum 1809.01671
  - Singular value statistics of two-point correlators 1902.11086

Other related works: 1801.03204, 1812.04770

Towards quantum gravity experiments using holography



◎ Not limited to classical limit

➔Several supporting evidences

e.g. check of the leading gravity correction for the black hole mass [M. Hanada, Y. Hyakutake, G. ishiki, and J. Nishimura, Science **344**, 882 (2013)]

Many "AdS/CMT" applications

This work:
approach quantum gravity by realizing corresponding non-gravity models in cold gases

Our proposal: coupled atom-molecule model [arXiv:1606.02454]

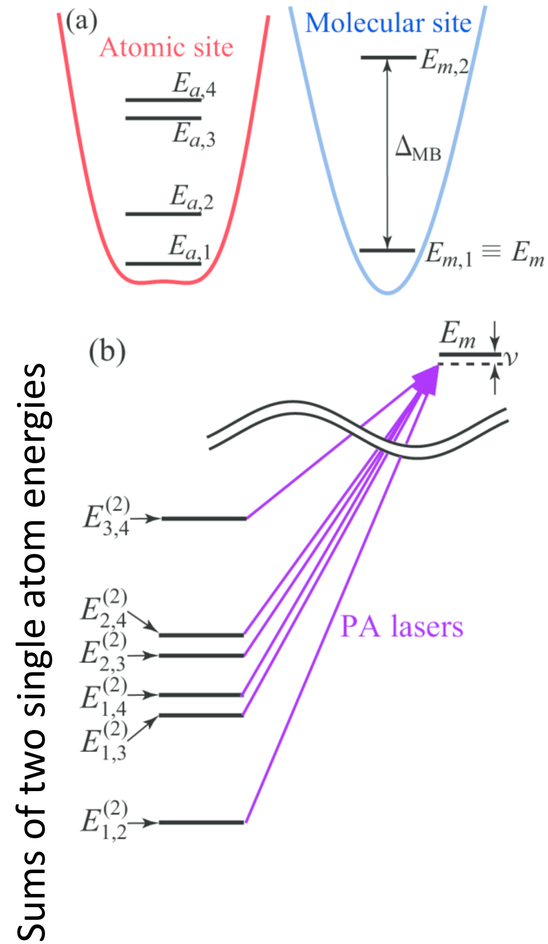
Consider atomic levels $i,j,\ldots = 1, 2, \ldots, N$ coupled to a molecule state m_1

$$\hat{H}_{\mathrm{m1}} = \nu \hat{m}^\dagger \hat{m} + \sum_{i,j} g_{ij} (\hat{m}^\dagger \hat{c}_j \hat{c}_i + h.c.)$$

$$g_{ij} = \frac{1}{2} \mathrm{sgn}(j-i) \int d\boldsymbol{r}\, \Omega_{i,j}(\mathbf{r}) w_m(\boldsymbol{r}) w_{a,i}(\boldsymbol{r})\, w_{a,j}(\boldsymbol{r})$$

Detuning ν: controlled by laser energy

$\Omega_{i,j}$: photoassociation (PA) laser
w_m: molecular site wavefunctions
$w_{a,i(j)}$: atomic site wavefunctions



$s = 1, 2, \ldots, n_s$

Consider multiple molecular states; assume they are short-lived
➔ integrate them out and obtain the effective model for atoms

$$\hat{H}_{\mathrm{eff}} = \sum_{s,i,j,k,l} \frac{g_{s,ij} g_{s,kl}}{\nu_s} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l$$

Realizing Dirac SYK model

$$\hat{H}_{\text{eff}} = \sum_{s,i,j,k,l} \frac{g_{s,ij} g_{s,kl}}{\nu_s} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l$$

$s = 1, 2, \ldots, n_s$

(Gaussian distribution for g)

(For simplicity we take $\nu_s = (-1)^s \sqrt{n_s} \sigma_s$)

approaches the real-coupling version of the Dirac SYK model as $n_s \rightarrow \infty$.

$$\hat{H} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ij;kl} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l,$$

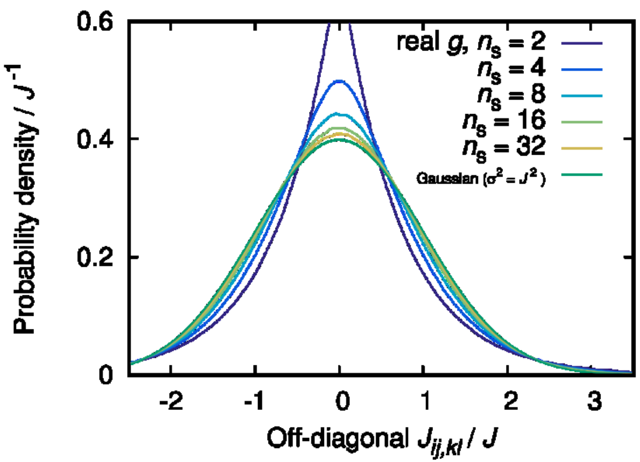
$$J_{ij;kl} = -J_{ji;kl} = -J_{ij;lk},$$

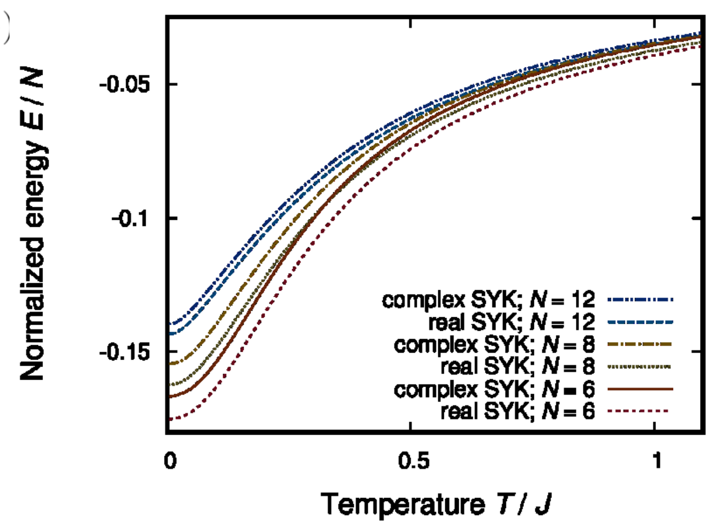
$$J_{ij;kl} = J_{kl,ij}$$

$$\overline{|J_{ij;kl}|^2} = \begin{cases} J^2 & (\{i,j\} \neq \{k,l\}) \\ 2J^2 & (\{i,j\} = \{k,l\}) \end{cases}$$

Real SYK:
Physical quantities coincide with those for complex SYK in $N \rightarrow \infty$ limit

Gaussian J reproduced





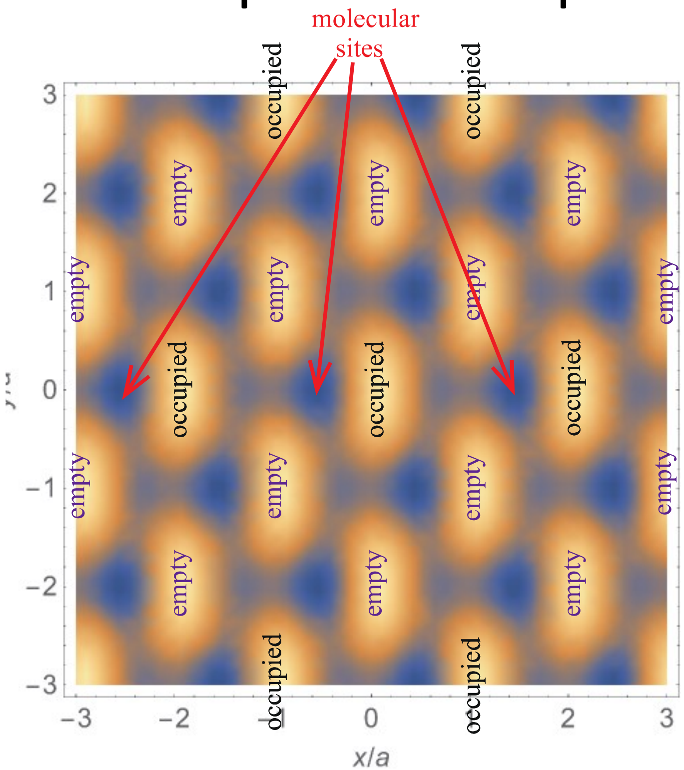
Optical lattice setup in our proposal

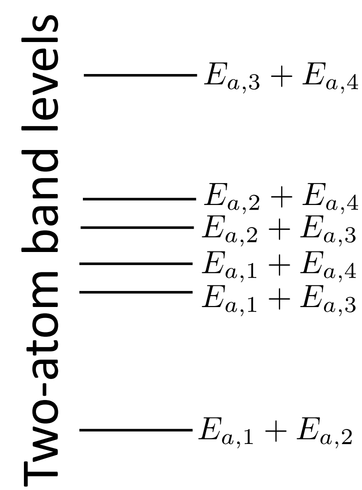
A double-well optical lattice

(no degeneracy in the band levels)

with [6]Li

(large recoil energy)





Possible to satisfy required conditions

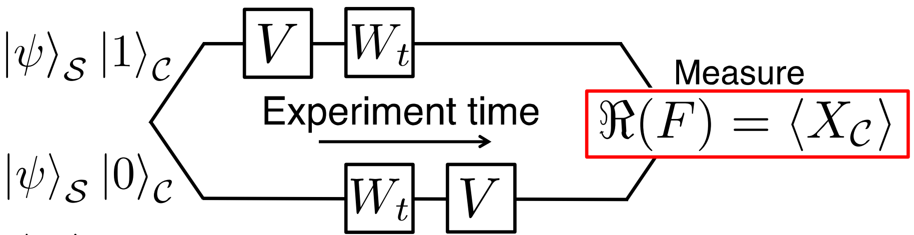
$$\max(t_i) \lesssim \hbar/\tau_{\rm exp} \ll J,$$
$$\max(\hbar\Gamma_{\rm PA}, \hbar\Gamma_{{\rm ms},s}) \ll |\nu_s| \ll \Delta_{\rm min}, \text{ for all } s,$$
$$\Delta_{\rm max} < \Delta_{\rm MB} < \tilde{\Delta},$$
$$|\nu_s| \ll |U_{s,s'}|, \text{ for all } s \text{ and } s',$$
$$|U_{s,s'}| < \Delta_{\rm min} \text{ or } \Delta_{\rm max} < |U_{s,s'}|, \text{ for all } s \text{ and } s'.$$

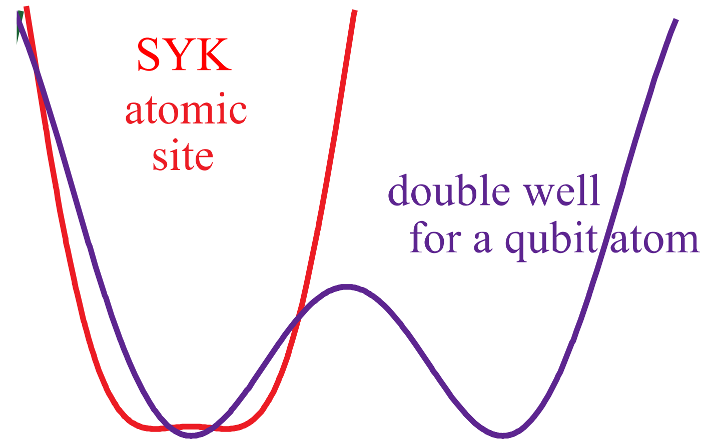
Danshita, Hanada, and MT, PTEP 2017, 083I01 (arXiv:1606.02454)

Out-of-time-order correlation measurement

Interferometric protocol proposed in
B. Swingle *et al.*: PRB **94**, 040302 (2016)

$$F(t) = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle.$$





$|\psi\rangle_S$: Initial state of the probed system

$|0\rangle_C, |1\rangle_C$: states of the control qubit

$$\widehat{W}(t) = \boxed{e^{iHt}}\,\widehat{W}e^{-iHt}$$

Time evolution with
$H' = -H$ ($\nu' = -\nu$)

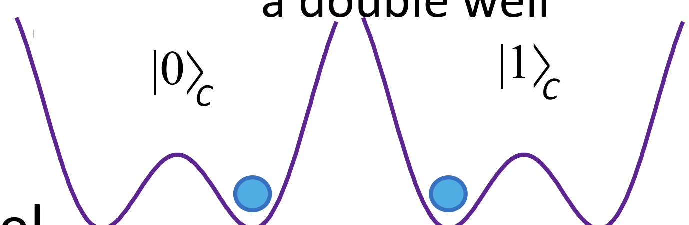
Create the cat state

$$|\Psi\rangle = \widehat{W}(t)\hat{V}|\psi\rangle_S|1\rangle_C + \hat{V}\widehat{W}(t)|\psi\rangle_S|0\rangle_C$$

by applying

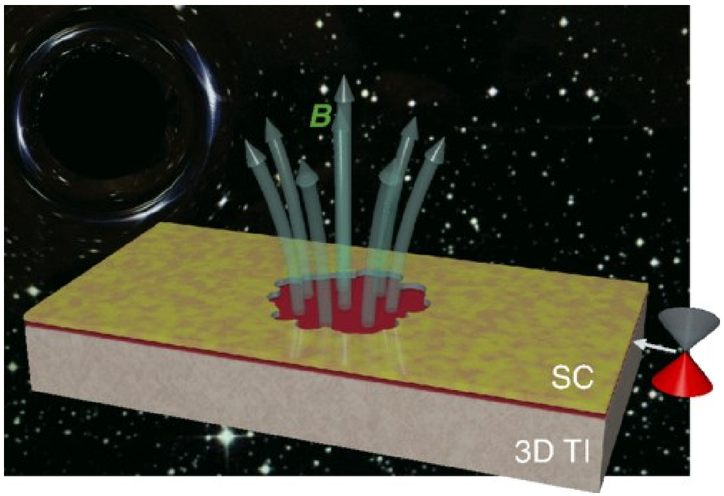
$\widehat{I_S} \otimes |0\rangle\langle 0|_C + \hat{V} \otimes |1\rangle\langle 1|_C$, $\widehat{W}(t) \otimes \widehat{I_C}$,
and $\hat{V} \otimes |0\rangle\langle 0|_C + \widehat{I_S} \otimes |1\rangle\langle 1|_C$ in this order, then
measure the qubit to find $\mathrm{Re}\,F(t)$ and $\mathrm{Im}\,F(t)$.

Our qubit *C*:
A single particle in
a double well



➔Implementation of this protocol in our model
using a qubit on additional optical double well [1606.02454]

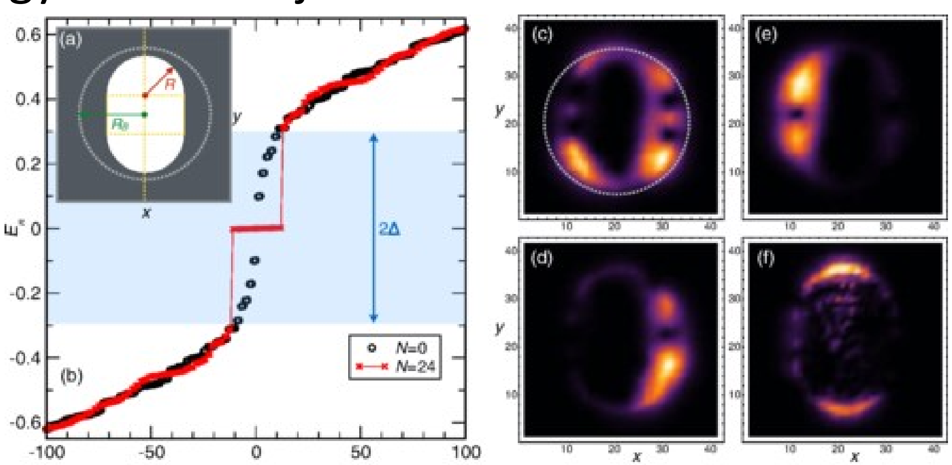
Proposals for experimental realization



N quanta of magnetic flux through a nanoscale hole

Inhomogeneous wave functions due to the irregular shape of the hole

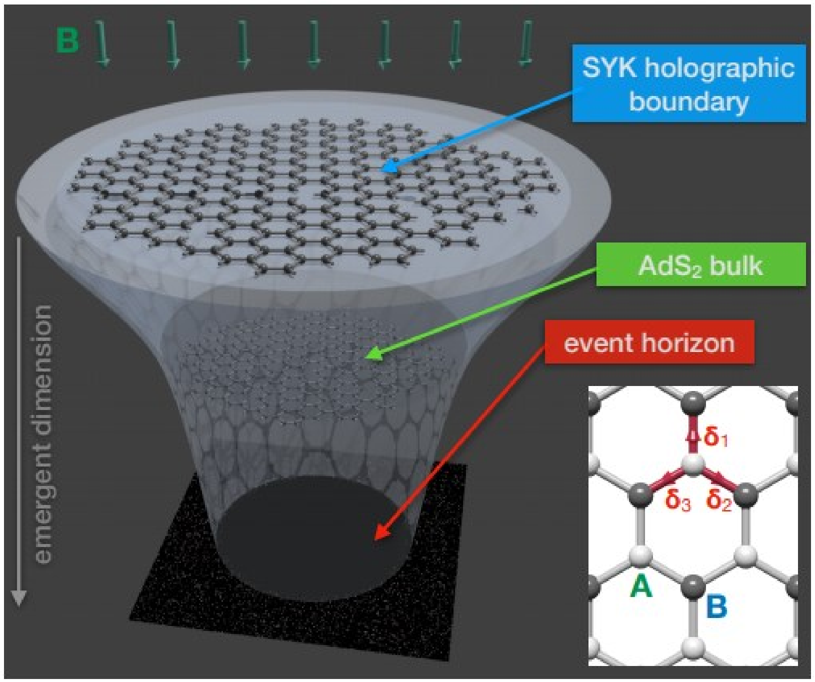
Zero energy states: Majorana fermions



D. I. Pikulin and M. Franz, "Black Hole on a Chip: Proposal for a Physical Realization of the Sachdev-Ye-Kitaev model in a Solid-State System", PRX **7**, 031006 (2017)

arXiv:1702.04426

Proposals for experimental realization



Anffany Chen, R. Ilan, F. de Juan, D.I. Pikulin, M. Franz,
"Quantum holography in a graphene flake with an irregular boundary",
PRL **121**, 036403 (2018) arXiv:1802.00802

Review Article | Published: 29 November 2018

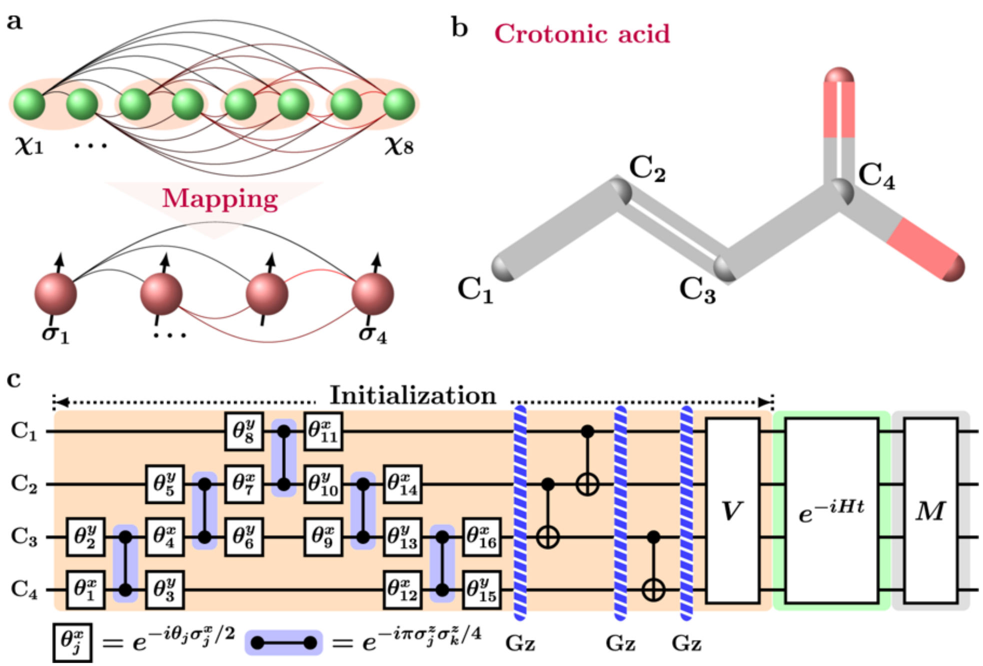
Mimicking black hole event horizons in atomic and solid-state systems

Marcel Franz & Moshe Rozali

Nature Reviews Materials **3**, 491–501 (2018) | Download Citation

NMR experiment for the SYK model

“Quantum simulation of the non-fermi-liquid state of Sachdev-Ye-Kitaev model” Zhihuang Luo, Yi-Zhuang You, Jun Li, Chao-Ming Jian, Dawei Lu, Cenke Xu, Bei Zeng and Raymond Laflamme, npj Quantum Information **5**, 53 (2019)

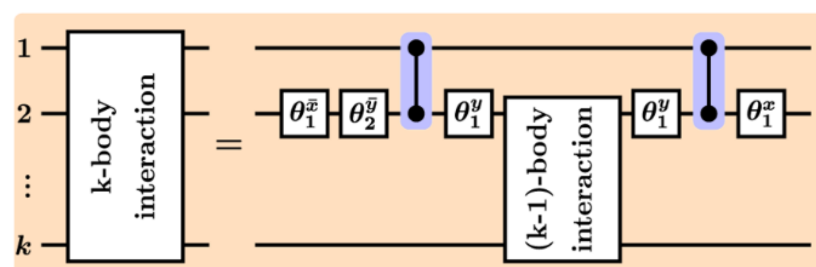


$$H = \frac{J_{ijkl}}{4!}\chi_i\chi_j\chi_k\chi_l + \frac{\mu}{4}C_{ij}C_{kl}\chi_i\chi_j\chi_k\chi_l$$

$$\chi_{2i-1} = \frac{1}{\sqrt{2}}\sigma_x^1\sigma_x^2\cdots\sigma_x^{i-1}\sigma_z^i,\ \chi_{2i} = \frac{1}{\sqrt{2}}\sigma_x^1\sigma_x^2\cdots\sigma_x^{i-1}\sigma_y^i.$$

$$H = \sum_{s=1}^{70} H_s = \sum_{s=1}^{70} a_{ijkl}^s \sigma_{\alpha_i}^1\sigma_{\alpha_j}^2\sigma_{\alpha_k}^3\sigma_{\alpha_l}^4$$

$$e^{-iH\tau} = \left(\prod_{s=1}^{70} e^{-iH_s\tau/n}\right)^n + \sum_{s<s'}\frac{[H_s, H_{s'}]\tau^2}{2n} + O(|a|^3\tau^3/n^2),$$



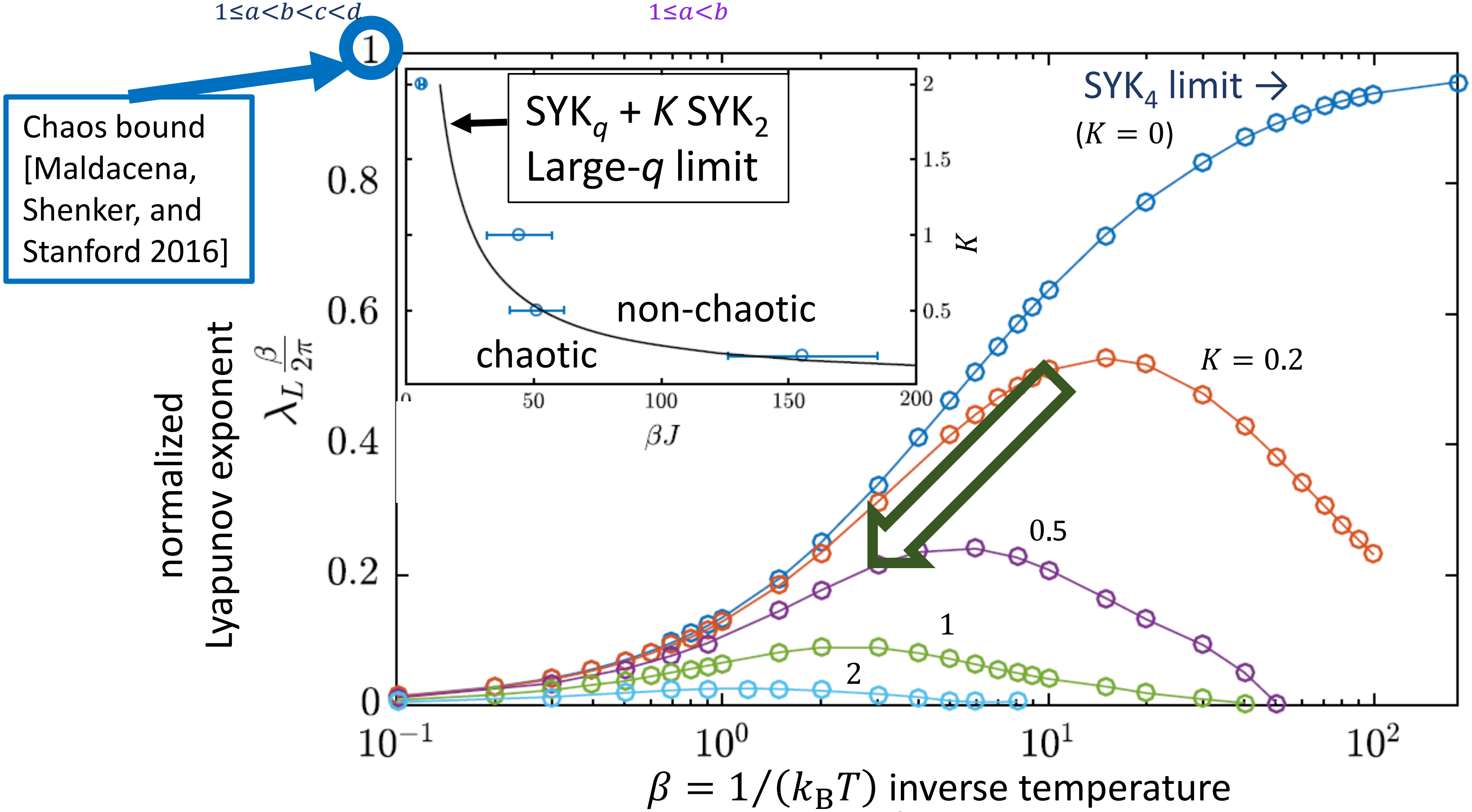
$SYK_4 + SYK_2$: Large-N calculation for OTOC

1707.02197

$$\hat{H} = \sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

SYK_4 SYK_2

K_{ab}: standard deviation $\frac{K}{\sqrt{N}}$



A. M. Garcia-Garcia, B. Loureiro, A. Romero-Bermudez, and MT, PRL **120**, 241603 (2018)

Deviation from the chaos bound as SYK_2 component is introduced

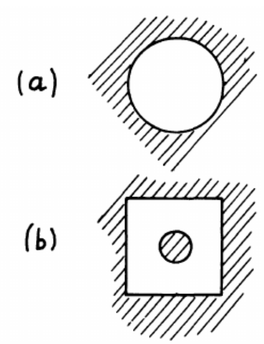
Contents

- ✓The Sachdev-Ye-Kitaev model
  - ✓Large-N solvability: conformal symmetry and maximal chaos
  - ✓Experimental proposal 1606.02454 (and realization)
  - ✓Deformation and suppression of maximal chaos 1707.02197
- Characterization of chaos in random systems 1702.06935
  - Quantum Lyapunov spectrum 1809.01671
  - Singular value statistics of two-point correlators 1902.11086

Other related works: 1801.03204, 1812.04770

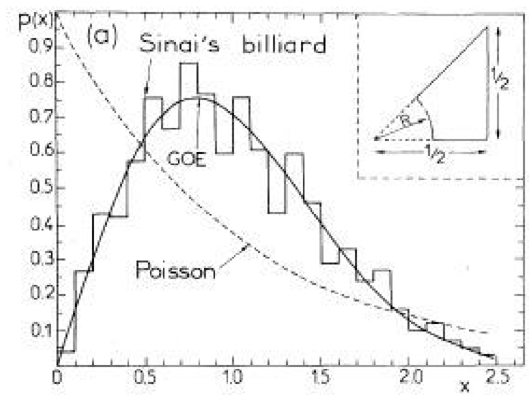
The Bohigas-Giannoni-Schmit conjecture

Assume quantum mechanical systems with a classical limit



circular: integrable

Sinai billiard: chaotic



"Spectral statistics of chaotic systems can be described as a random matrix"

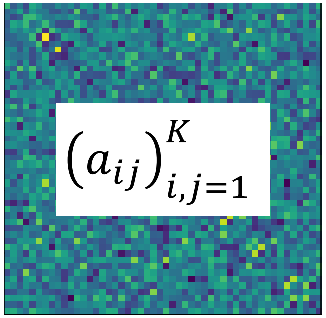
O. Bohigas, M. J. Giannoni, and C. Schmit,
Phys. Rev. Lett. 52, 1 (1984);
J. de Phys. Lett. 45, 1015 (1984).

Justifications:

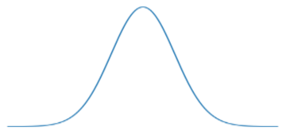
Non-linear sigma-model (Andreev 1993, Altland 2015)
Gutzwiller trace formula in terms of periodic orbits (Berry 1985, Gutzwiller 1990, Sieber, Richter, Braun, Muller, Heusler, …)

Also more examples including systems without clear classical version

Gaussian random matrices



$a_{ij} = a_{ji}^*$



Gaussian distribution

$$\text{Density} \propto e^{-\frac{\beta K}{4}\mathrm{Tr}H^2} = \exp\left(-\frac{\beta K}{4}\sum_{i,j}^{K}|a_{ij}|^2\right)$$

Real (β=1): Gaussian Orthogonal Ensemble (GOE)

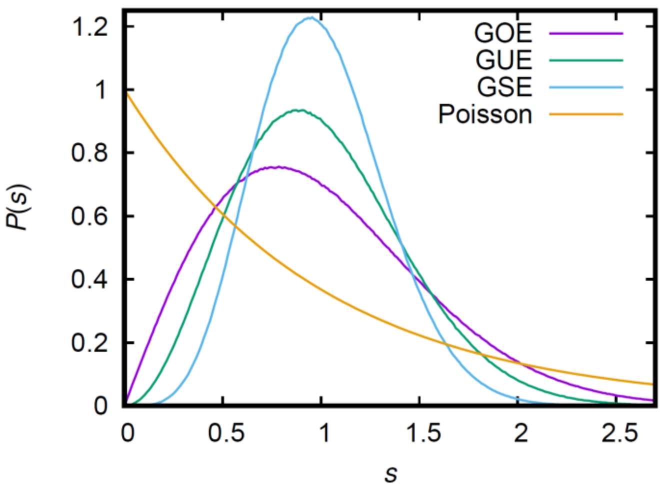
Complex (β=2): G. Unitary E. (GUE)

Quaternion (β=4): G. Symplectic E. (GSE)

Joint distribution function for eigenvalues $\{e_j\}$

Level repulsion

$$p(e_1, e_2, \ldots, e_K) \propto \prod_{1\le i<j\le K} |e_i - e_j|^{\beta} \prod_{i=1}^{K} e^{-\beta K e_i{}^2/4}$$



- $P(s)$: Distribution of normalized level separation

$$s_j = \frac{e_{j+1} - e_j}{\Delta(\bar{e})}$$

GOE/GUE/GSE: $P(s) \propto s^{\beta}$ at small s, has e^{-s^2} tail

Uncorrelated (Poisson): $P(s) = e^{-s}$

- $\langle r \rangle$: Average of neighboring gap ratio

$$r = \frac{\min(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ \ e_{i+2} - e_{i+1})}$$

	Uncorrelated	GOE	GUE	GSE
$\langle r \rangle$	2log 2 – 1 = 0.38629...	0.5307(1)	0.5996(1)	0.6744(1)

[Y. Y. Atas *et al.* PRL 2013]

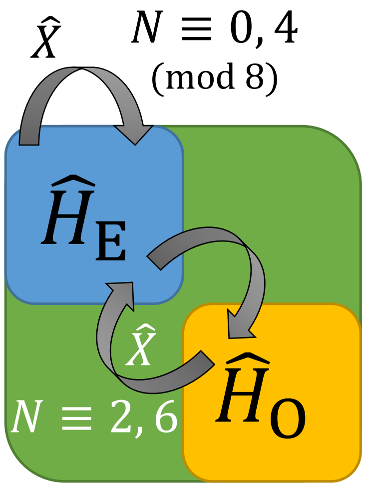
N mod 8 classification of Majorana SYK$_{q=4}$

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a<b<c<d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

SPT phase classification for class BDI:
$\mathbb{Z} \rightarrow \mathbb{Z}_8$ due to interaction
[L. Fidkowski and A. Kitaev, PRB 2010, PRB 2011]

Introduce $N/2$ complex fermions $\hat{c}_j = \frac{(\hat{\chi}_{2j-1} + i\hat{\chi}_{2j})}{\sqrt{2}}$

$\hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$ respects the complex fermion parity
Even ($\hat{H}_{\mathrm{E}}$) and odd ($\hat{H}_{\mathrm{O}}$) sectors: $L = 2^{N/2-1}$ dimensions



N mod 8	0	2	4	6
η	-1	+1	+1	-1
$\hat{X}^2$	**+1**	+1	**-1**	-1
$\hat{X}$ maps H_{E} to	H_{E}	H_{O}	H_{E}	H_{O}
Class	**AI**	**A+A**	**AII**	**A+A**
Gaussian ensemble	**GOE**	**GUE**	**GSE**	**GUE**

$$\hat{X} = \hat{K} \prod_{j=1}^{N/2} (\hat{c}_j^\dagger + \hat{c}_j)$$

$$\hat{X} \hat{c}_j \hat{X} = \eta \hat{c}_j^\dagger; \; [\hat{X}, \hat{H}] = 0$$

[You, Ludwig, and Xu, PRB 2017]

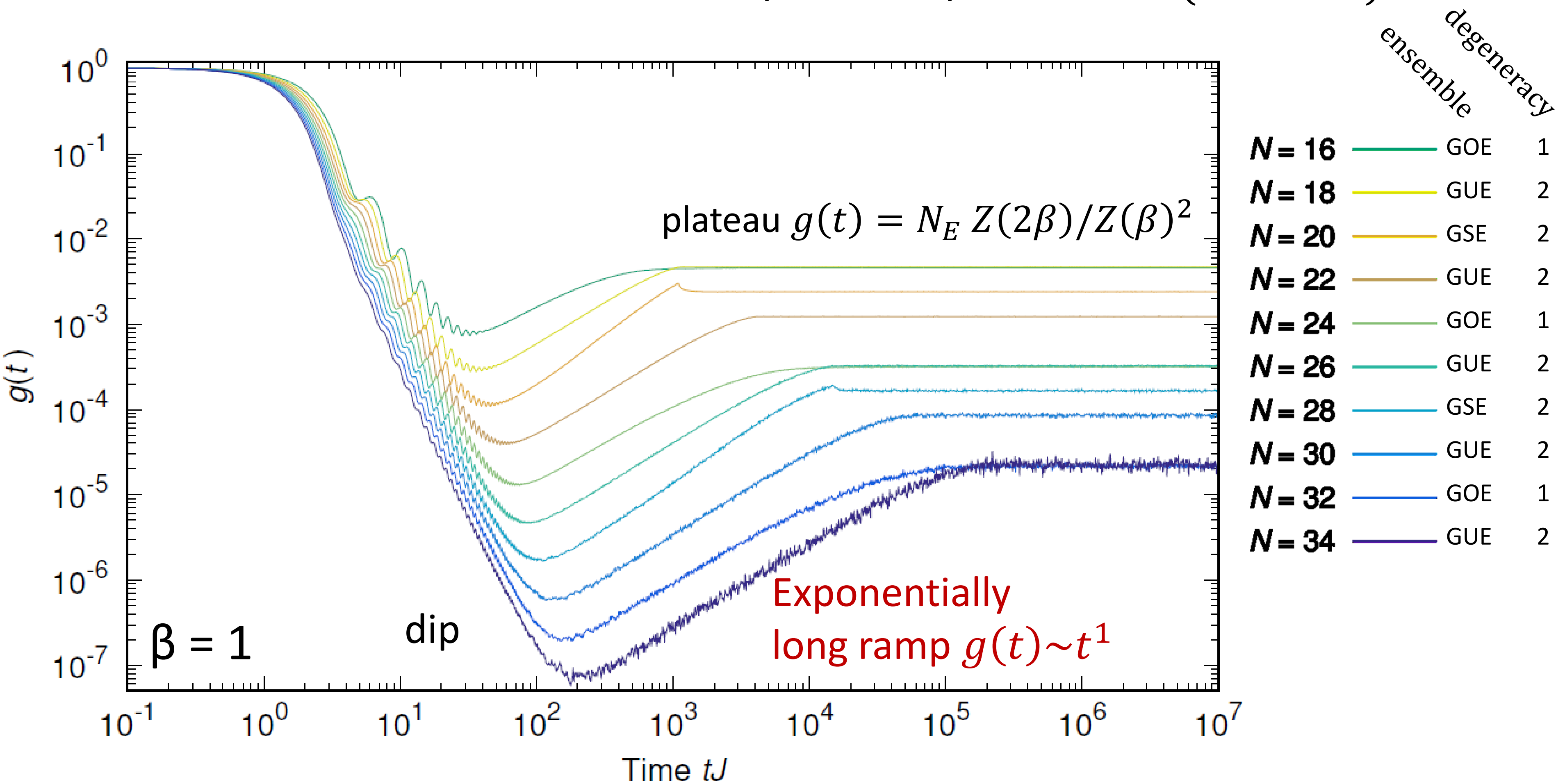
[Fadi Sun and Jinwu Ye, 1905.07694]
for SYK$_q$, supersymmetric SYK

SYK: sparse matrix, but energy spectral statistics strongly resemble that of the corresponding (dense) Gaussian ensemble [Cotler, …, MT, JHEP 2017]

The spectral form factor $g(\beta,t)=\frac{\langle |Z(\beta,t)|^2\rangle_J}{\langle Z(\beta)\rangle_J{}^2}$

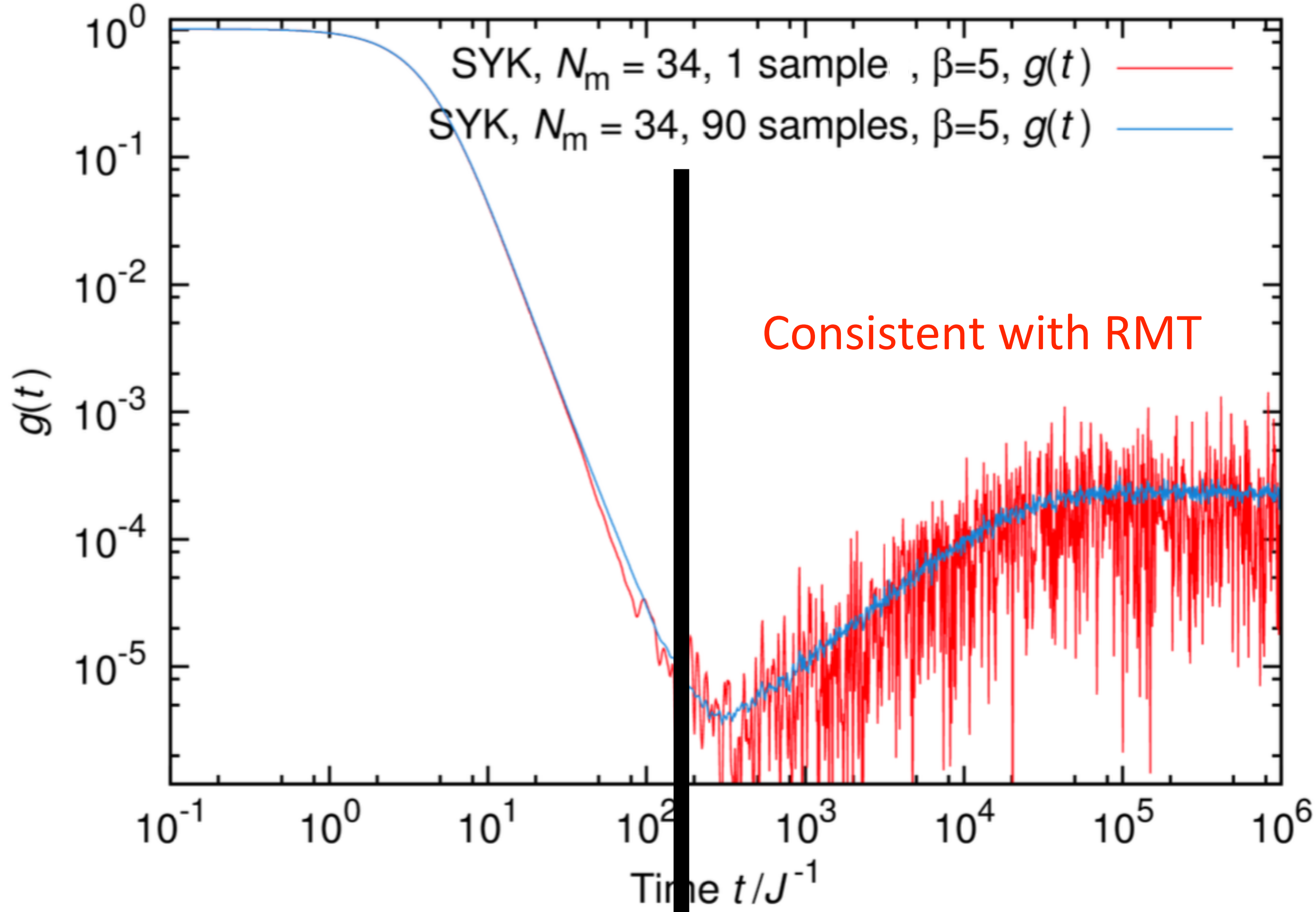
1611.04650

$$Z(\beta,t)=Z(\beta+\mathrm{i}t)=\mathrm{Tr}\left(\mathrm{e}^{-\beta\hat{H}-\mathrm{i}\hat{H}t}\right)$$



(Non-)self-averaging

1611.04650



1 sample = many samples
Self-averaging

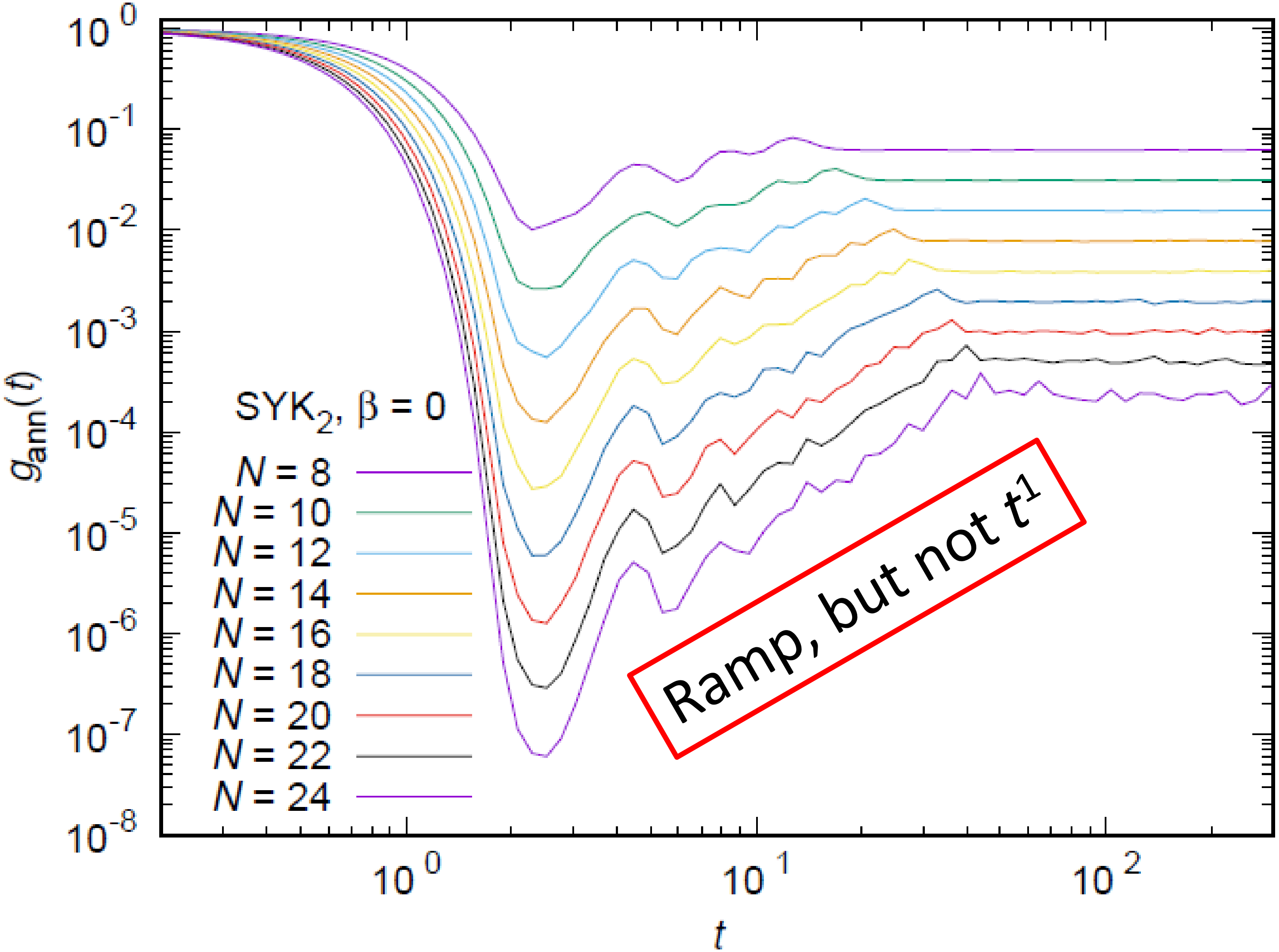
"Gravity"
$1/N$ expansion

$O(1)$ variance for 1 sample
Not self-averaging

"Random matrix theory"
$1/K \sim e^{-N}$ expansion

$K = 2^{N/2}$

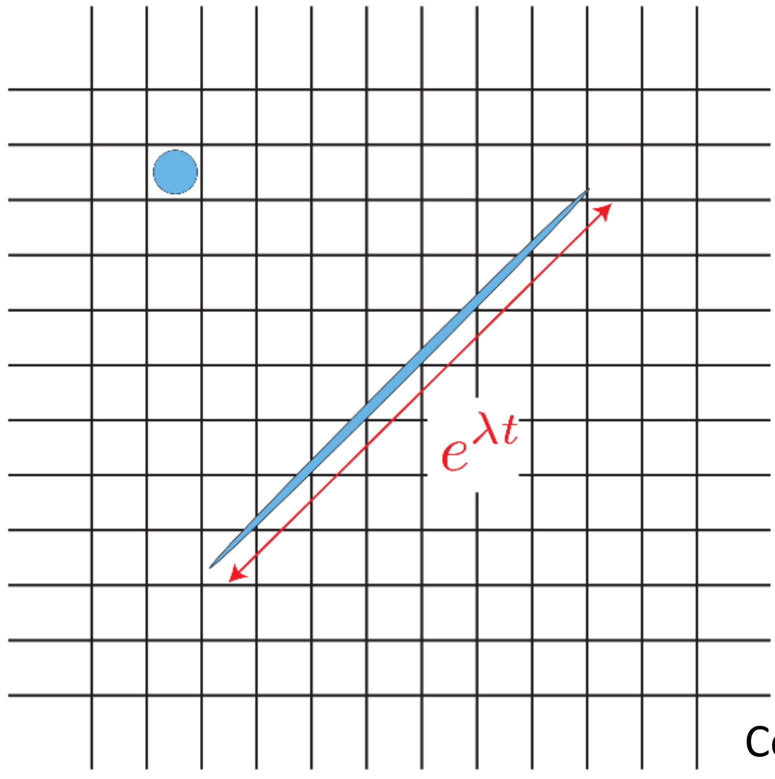
Note: dip-ramp-plateau structure does not require chaos



"Randomness and Chaos in Qubit Models"

Pak Hang Chris Lau, Chen-Te Ma, Jeff Murugan, and MT, Phys. Lett. B **795**, 230 (2019).

Lyapunov growth of phase space



Coarse-grained

- Just one direction?
- If more than one, what are relations between λ?

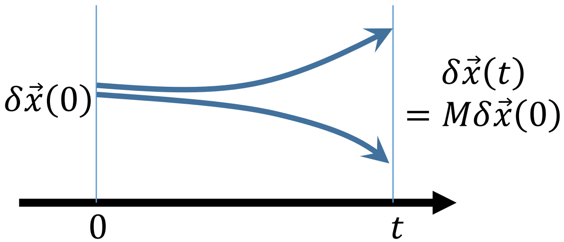
1809.01671

Quantum Lyapunov spectrum

Finite-time **classical Lyapunov spectrum**: obeys RMT statistics for chaos

[Hanada, Shimada, and MT: PRE **97**, 022224 (2018)]

Singular values of $M_{ij} = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)$ **at finite *t***: $\{s_k(t)\} = \{e^{\lambda_k t}\}$



$L = \{x_i(t), p_j(0)\}_{\mathrm{PB}}^2 = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)^2 \to e^{2\lambda_{\mathrm{L}} t}$ for large t

OTOC: $C_T(t) = \left\langle \left| [\hat{W}(t), \hat{V}(t=0)] \right|^2 \right\rangle = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle + \cdots$

Quantum Lyapunov spectrum: Define $\hat{M}_{ab}(t)$ as (anti)commutator of $\hat{O}_a(t)$ and $\hat{O}_b(0)$

$$\hat{L}_{ab}(t) = \left[\hat{M}(t)^\dagger \hat{M}(t)\right]_{ab} = \sum_{j=1}^{N} \hat{M}_{ja}(t)^\dagger \hat{M}_{jb}(t)$$

For $N \times N$ matrix $\langle \phi | \hat{L}_{ab}(t) | \phi \rangle$, obtain singular values $\{s_k(t)\}_{k=1}^N$.

The Lyapunov spectrum is defined as $\left\{\lambda_k(t) = \frac{\log s_k(t)}{2t}\right\}$.

Quantum Lyapunov spectrum for SYK model + modification

$$\hat{H} = \sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

J_{abcd}: s. d. $= \frac{\sqrt{6}}{N^{3/2}}$

K_{ab}: s. d. $= \frac{K}{\sqrt{N}}$

- Define $\hat{L}_{ab}(t) = \sum_{j=1}^{N} \widehat{M}_{ja}(t)\widehat{M}_{jb}(t)$ for time-dependent anticommutator $\widehat{M}_{ab}(t) = \{\hat{\chi}_a(t), \hat{\chi}_b(0)\}$.

- Obtain the singular values $\{a_k(t)\}_{k=1}^{K}$ of $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$

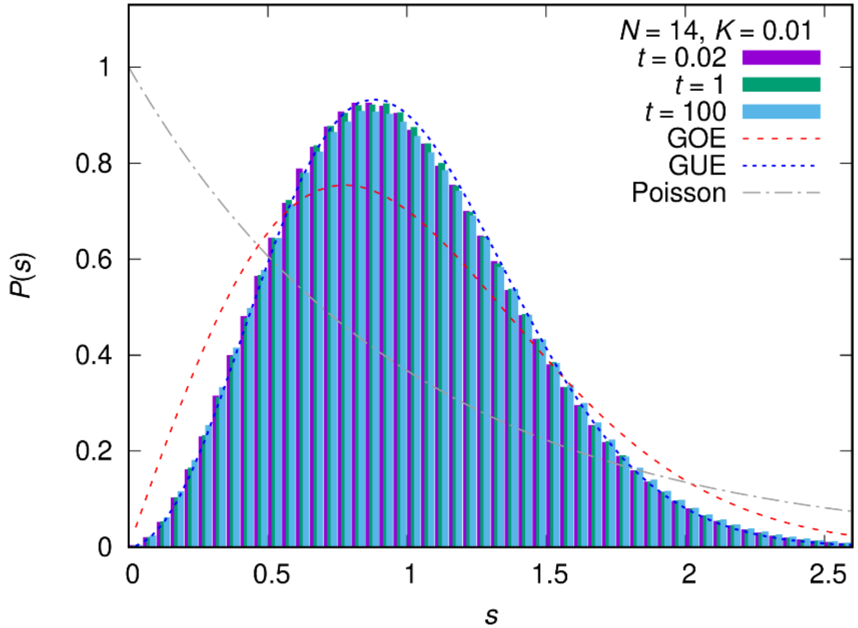
- Quantum Lyapunov spectrum: $\left\{\lambda_k(t) = \frac{\log a_k(t)}{2t}\right\}_{k=1,2,\ldots,K}$

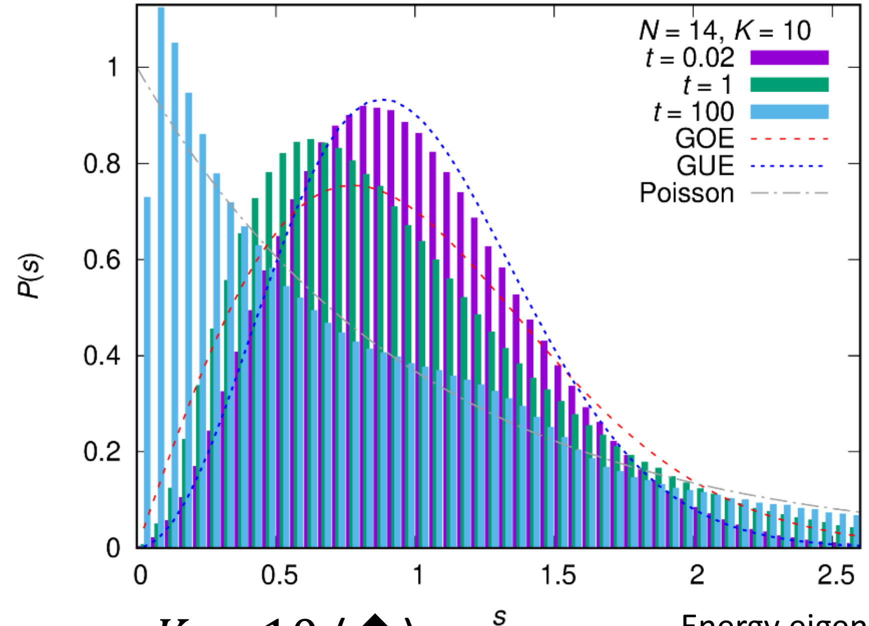
  (also dependent on state ϕ)

Other possibilities: see Rozenbaum-Ganeshan-Galitski, 1801.10591; Hallam-Morley-Green: 1806.05204

Spectral statistics of quantum Lyapunov spectrum: SYK

1809.01671

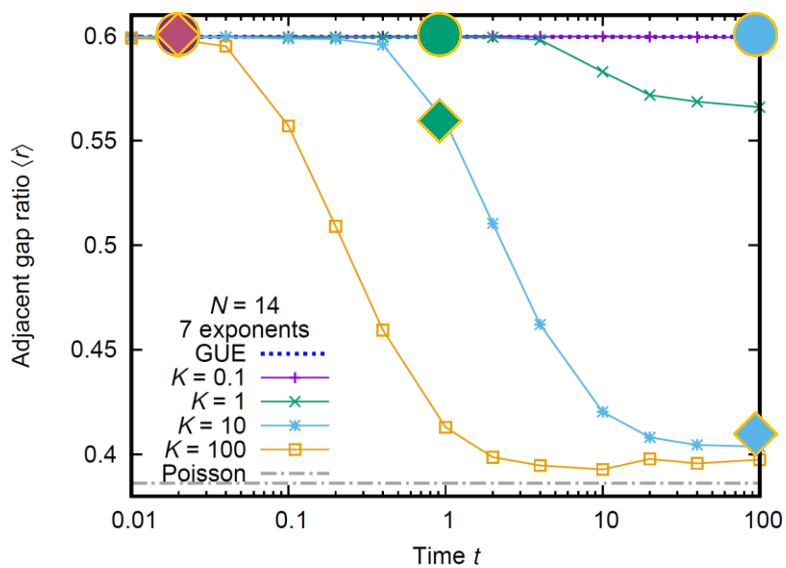




$K = 0.01$ (●): Remains GUE for long time

$K = 10$ (◆): Approaches Poisson

Energy eigenstates
$N/2$ larger exponents

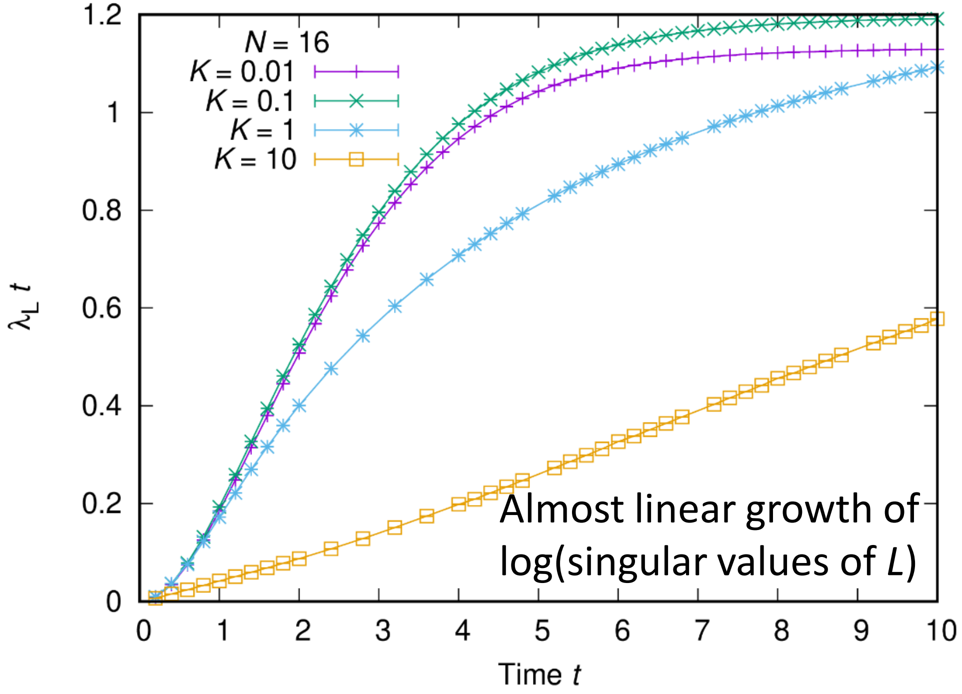


$\langle r \rangle$: average of $\dfrac{\min(s_i, s_{i+1})}{\max(s_i, s_{i+1})}$

(fixed-i unfolding: unfold each gap $g_i = \lambda_{i+1} - \lambda_i$ using its average $\langle g_i \rangle_J$, $s_i = g_i / \langle g_i \rangle_J$)

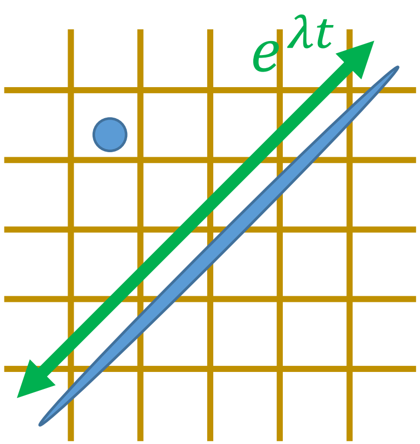
1809.01671

Growth of (largest Lyapunov exponent)*time



Kolmogorov-Sinai entropy vs entanglement entropy production

Coarse-grained entropy
= log(# of cells covering the region)
~ (sum of positive λ) t



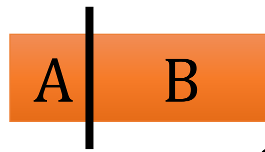
Kolmogorov-Sinai entropy h_{KS}
= (sum of positive λ)
= entropy production rate

Initial state with $S_{\mathrm{EE}} = 0$:

$$|\psi(t=0)\rangle = |000 \ldots 000\rangle$$

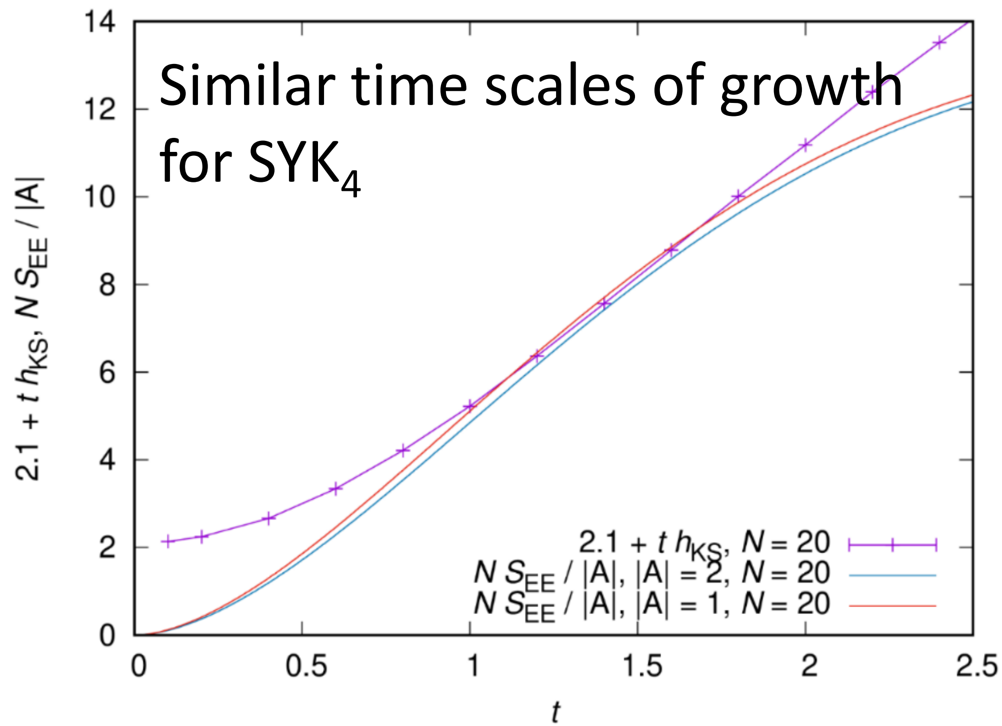
in the complex fermion basis

$$\hat{c}_j = \frac{(\chi_{2j-1} + \mathrm{i}\chi_{2j})}{\sqrt{2}}$$



$\rho_{\mathrm{A}}(t) = \mathrm{Tr}_{\mathrm{B}}\, \rho(t)$, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$

$$S_{\mathrm{EE}}(t) = -\mathrm{Tr}\, \rho_{\mathrm{A}}(t) \log\big(\rho_{\mathrm{A}}(t)\big)$$



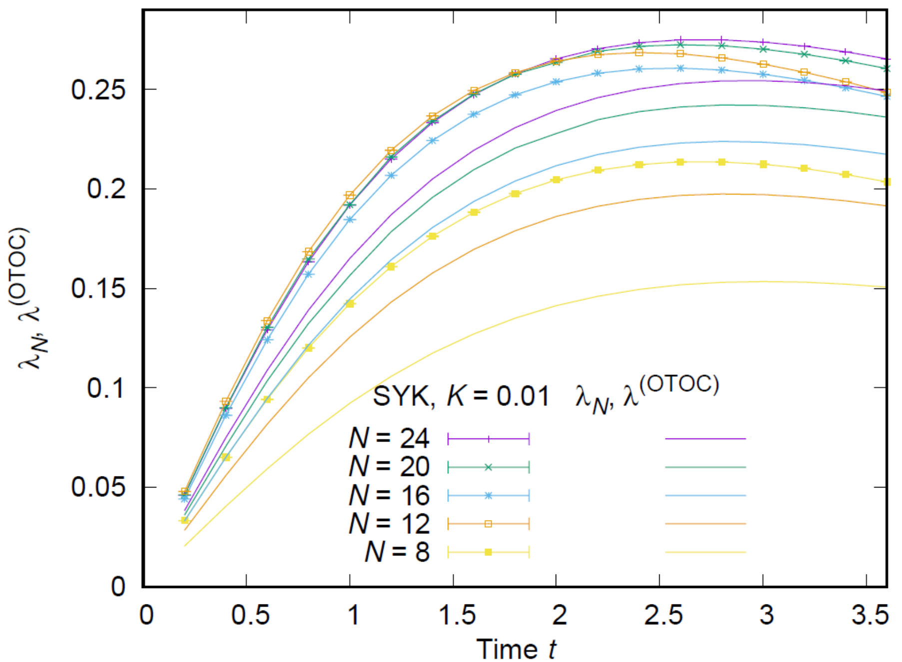
1809.01671

Fastest entropy production?

SYK$_4$ limit

- λ_N and $\lambda_{\text{OTOC}} = \frac{1}{2t}\log\left(\frac{1}{N}\sum_{i=1}^{N} e^{2\lambda_i t}\right)$ approach each other; difference decreases as $1/N$
- Same for λ_N and λ_1:

  all exponent → single peak
- All saturate the MSS bound at strong coupling (low *T*) limit
- Growth rate of entanglement entropy $\sim h_{\text{KS}}$ = sum of positive (all) λ_i



→ [conjecture] SYK model: not only the fastest scramblers, but also fastest entropy generators

QLS: The case of the random field XXZ model

$$\hat{H} = \sum_{i}^{N} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1} + \sum_{i}^{N} h_i \widehat{S_i^z}$$

h_i: uniform distribution $[-W, W]$

Many-body localization (MBL) transition at $W = W_c \sim 3.5$

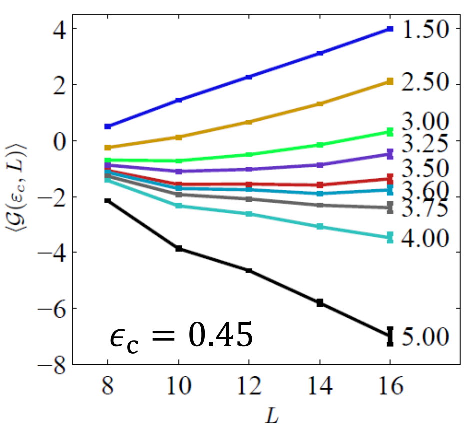
(though recently disputed; e.g. $W_c \geq 5$ proposed in E. V. H. Doggen et al., [1807.05051] using large systems with time-dependent variational principle & machine learning)

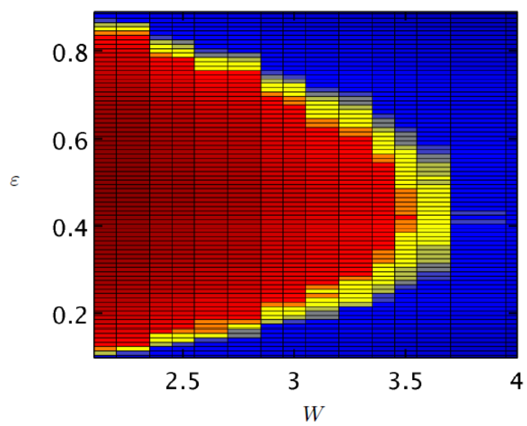
e.g. M. Serbyn, Z. Papic, and D. A. Abanin,
Phys. Rev. X **5**, 041047 (2015) (arXiv:1507.01635)

Matrix element of local perturbation

$$\mathcal{G}(\varepsilon, L) = \ln \frac{|V_{n,n+1}|}{E'_{n+1} - E'_n}$$

Energy separation of neighboring energy eigenstates





cf. MBL in short-range SYK [García-García and MT, Phys. Rev. B **99**, 054202 (2019)]; Localization of fermions on quasiperiodic lattice with attractive on-site interaction [Phys. Rev. A **82**, 043613 (2010)]

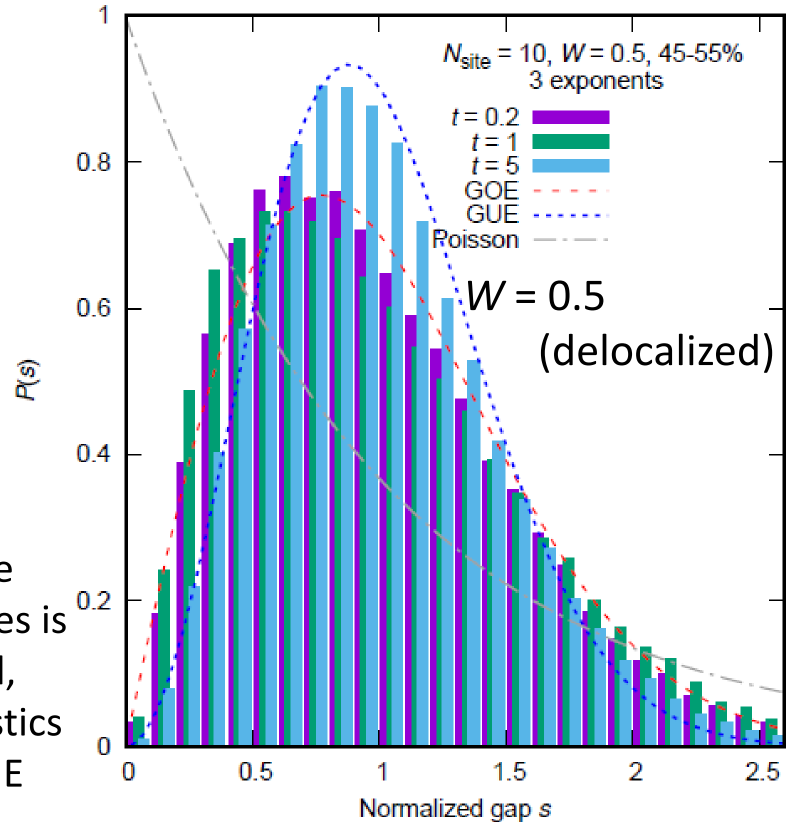
1809.01671

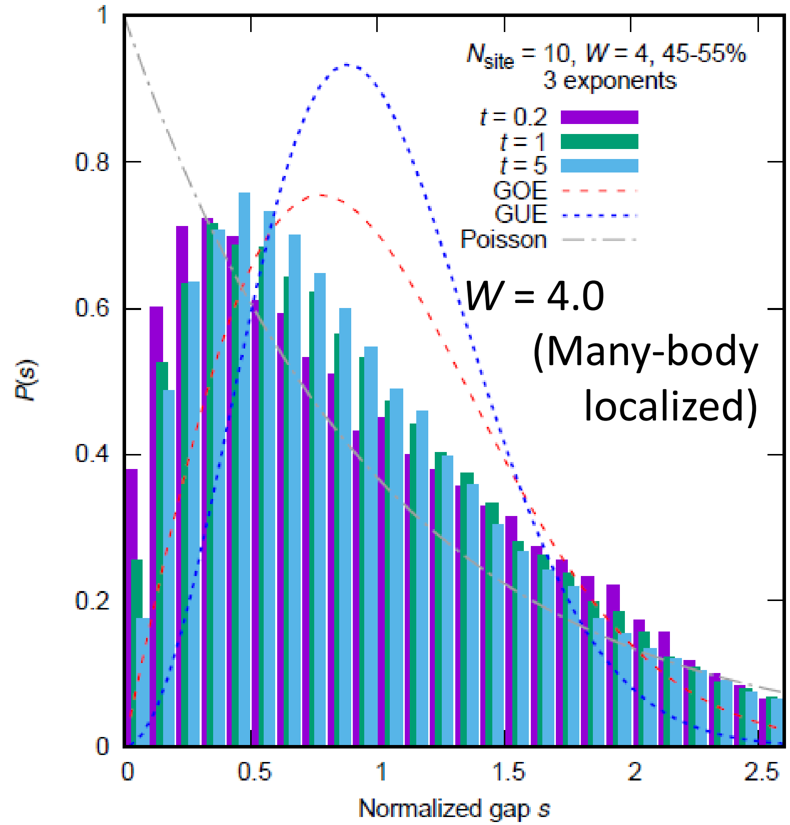
Spectral statistics of QLS for random field XXZ

$$\hat{H} = \sum_{i}^{N} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1} + \sum_{i}^{N} h_i \widehat{S_i^z}$$

h_i: uniform distribution $[-W, W]$

$$\hat{M}_{ab}(t) = [\widehat{S_a^+}(t), \widehat{S_b^-}(0)]$$





- Exponential growth of the singular values is not observed, but the statistics approach GUE

Quantum Lyapunov spectrum distinguishes chaotic and non-chaotic phases

Two-point correlation function

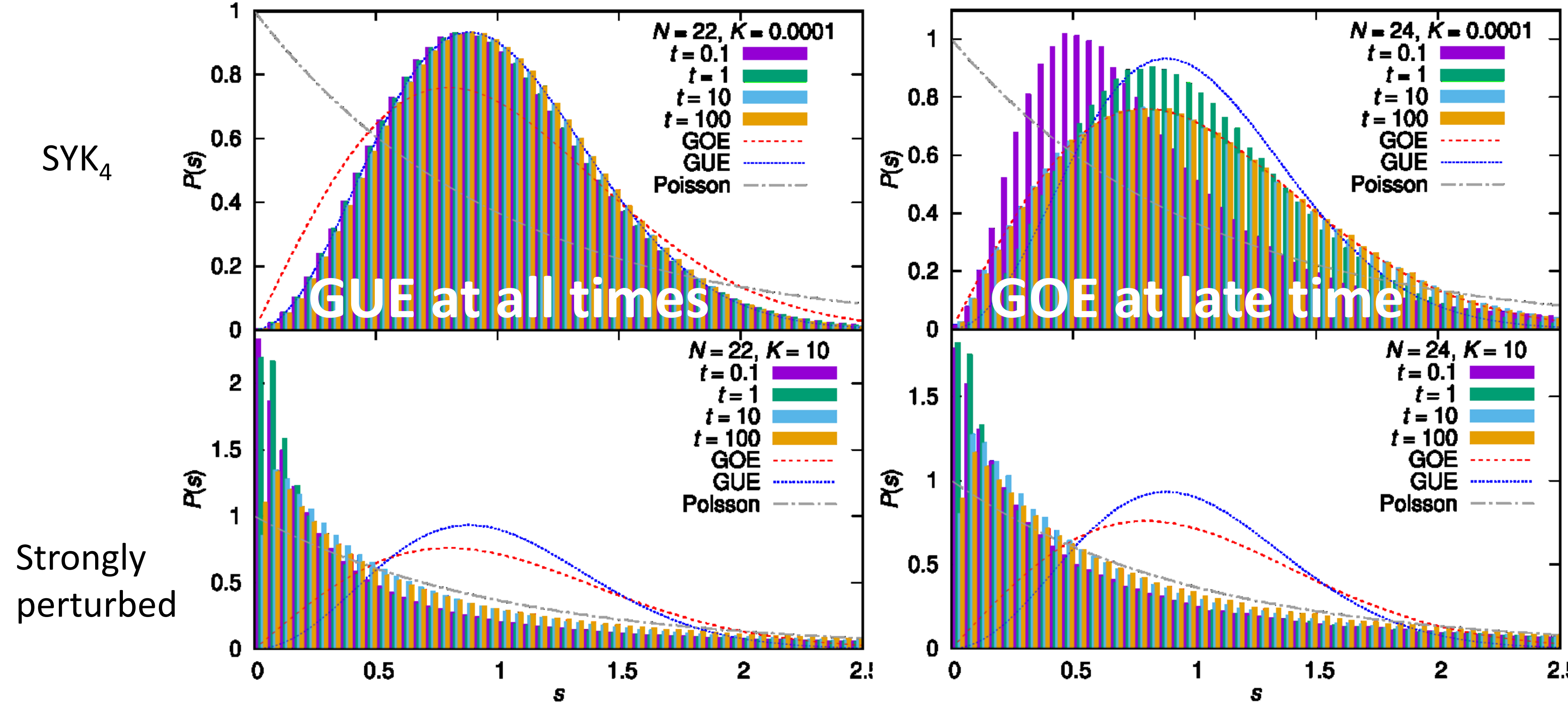
$$G_{ab}^{(\phi)}(t) = \langle \phi | \hat{\chi}_a(t) \hat{\chi}_b(0) | \phi \rangle$$

Singular value statistics of **two-point time correlators**

1902.11086

$$G^{(\phi)}_{ab}(t) = \langle\phi|\hat{\chi}_a(t)\hat{\chi}_b(0)|\phi\rangle \text{ as a matrix}$$

$$\lambda_j(t) = \log\left[\text{singular values of}\left(G^{(\phi)}_{ab}(t)\right)\right]$$



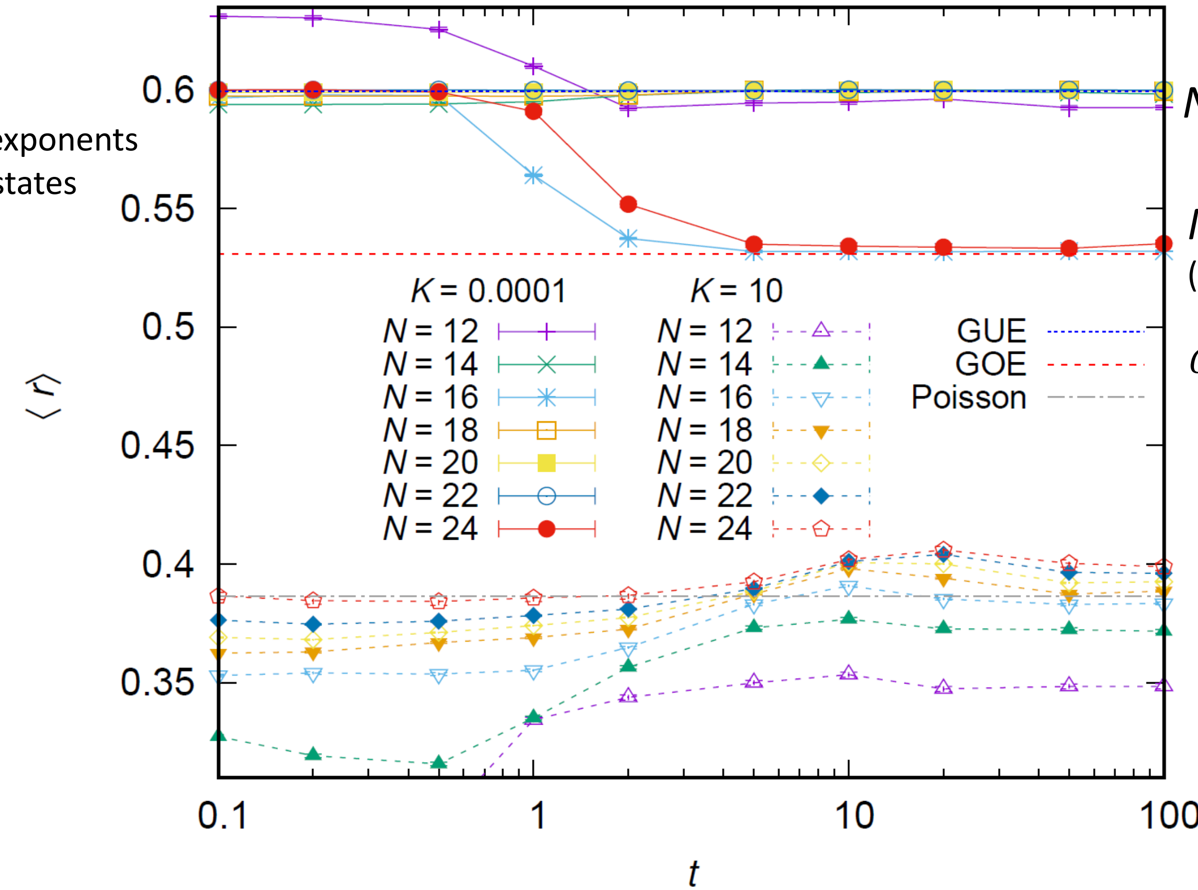
1902.11086

$\langle r \rangle$: average of the adjacent gap ratio $\frac{\min(\lambda_{i+1}-\lambda_i\,,\lambda_{i+2}-\lambda_{i+1})}{\max(\lambda_{i+1}-\lambda_i\,,\lambda_{i+2}-\lambda_{i+1})}$

Uncorrelated (Poisson): $2\log 2 - 1 \approx 0.386$

Correlated: larger (GOE: 0.5307, GUE: 0.5996 etc.) [Atas *et al.*, PRL 2013]

SYK, larger $N/2$ exponents
ϕ: energy eigenstates
fixed-i unfolded



N mod 8 = 2, 4, 6: GUE

N mod 8 = 0: GOE
(the matrix is symmetric)

$G_{ab}^{(\phi)} = \langle\phi|\hat{\chi}_a(t)\hat{\chi}_b(0)|\phi\rangle$

At late time, for two-point correlator singular values,
Random matrix behavior ⇔ chaotic

Summary

$$\hat{H}_{\mathrm{SYK}} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

black hole

holography

non-gravitational quantum system

cold atom realization

Danshita, Hanada and MT, PTEP 2017 [1606.02454]

maximally chaotic (chaos bound)

maximally chaotic (chaos bound)

$\lambda_{\mathrm{Lyapunov}}$

random matrix behavior of finite-time Lyapunov spectrum

[Quantum] Gharibyan, Hanada, Swingle and MT, JHEP 2019 [1809.01671]

modifications to study chaos / integrable transition & many-body localization

García-García et al., PRL 2018 [1707.02197]

random matrix behavior of two-point correlators

Gharibyan, Hanada, Swingle and MT, 1902.11086

P. H. C. Lau, **Chen-Te Ma**, J. Murugan, and MT, Phys. Lett. B 2019 [1812.04770]