# Characterization of many-body quantum chaos by quantum Lyapunov spectrum and two-point correlators

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**Motivation**

Frequently used criteria for characterizing quantum chaos:

- Fine-grained energy level statistics (described by random matrix theory (RMT))
- Out-of-time-order correlators (OTOC) (exponential Lyapunov growth)

- Relation between the two criteria?
- Relation to classical chaos?

We try to characterize quantum many-body chaos by

- Generalizing OTOC to define Lyapunov spectrum
- Considering simpler quantities more accessible to experiment

## Quantum Lyapunov spectrum (arXiv:1809.01671)

Finite-time **classical Lyapunov spectrum**: obeys RMT statistics for chaos

[M. Hanada, H. Shimada, and MT: PRE **97**, 022224 (2018)]

Singular values of $\left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)$ **at finite** $t$: $\{s_k(t)\} = \{e^{\lambda_k t}\}$

$$L = \{x_i(t), p_j(0)\}_{\mathrm{PB}}{}^2 = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)^2 \to e^{2\lambda_L t} \text{ at large } t$$

OTOC: $C_T(t) = \left\langle \left| [\hat{W}(t), \hat{V}(t=0)] \right|^2 \right\rangle = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle + \cdots$

**Quantum Lyapunov spectrum:**

Define $\hat{M}_{ab}(t)$ as (anti)commutator of $\hat{O}_a(t)$ and $\hat{O}_b(0)$

$$\hat{L}_{ab}(t) = \sum_{j=1}^{N} \hat{M}_{ja}(t)^\dagger \hat{M}_{jb}(t)$$

For $N \times N$ matrix $\langle \phi | \hat{L}_{ab}(t) | \phi \rangle$, obtain singular values $\{s_k(t)\}_{k=1}^{N}$.

The Lyapunov spectrum is defined as $\left\{\lambda_k(t) = \frac{\log s_k(t)}{2t}\right\}$.

**Fixed-*i* unfolding**

Divide $i$-th gap $g_i = \lambda_{k+1} - \lambda_k$ by its sample average $\Delta_i$ to obtain unfolded gap $s_i$

➔ gap distribution $P(s)$, adjacent gap ratio $r = \frac{\min(s_{i+1}-s_i, s_{i+2}-s_{i+1})}{\max(s_{i+1}-s_i, s_{i+2}-s_{i+1})}$

### SYK model

$$\hat{H} = \sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

SYK$_4$, SYK$_2$

$J_{abcd}$: average 0, standard deviation $\frac{\sqrt{6}J}{N^{3/2}}$ $K_{ab}$: average 0, standard deviation $\frac{K}{\sqrt{N}}$

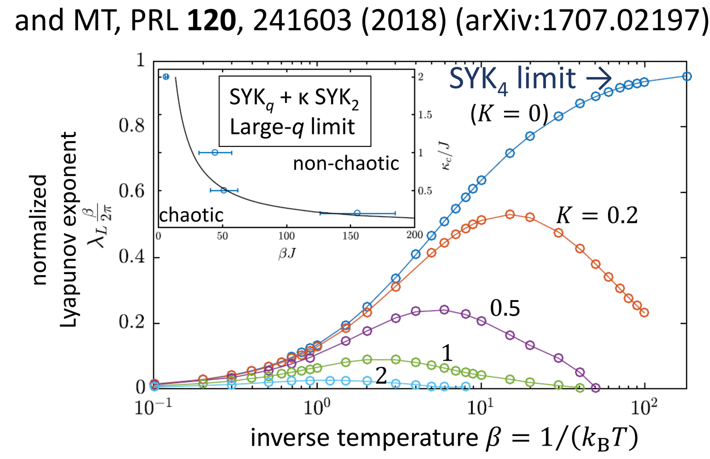
solvable in large-$N$ limit, strongly coupled at low $T$

SYK$_4$ limit: maximally chaotic

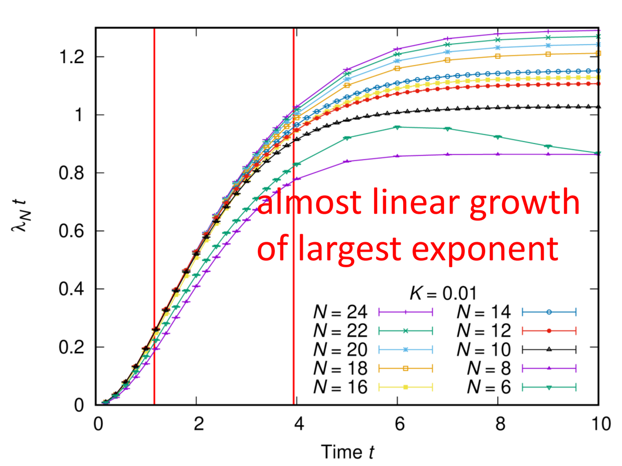
(satisfies Maldacena-Shenker-Stanford chaos bound)

Random hopping term suppresses chaos

[A. M. García-García, B. Loureiro, A. Romero-Bermudez, and MT, PRL **120**, 241603 (2018) (arXiv:1707.02197)]



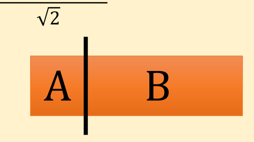

$\hat{M}_{ab}(t) = \{\hat{\chi}_a(t), \hat{\chi}_b(0)\}$



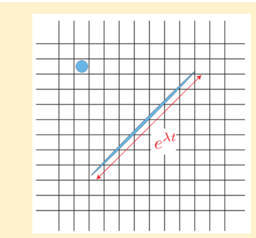

**Comparison with entanglement entropy**

For initial state $|\psi(t=0)\rangle = |\underbrace{000\ldots000}_{N/2}\rangle$ (complex fermion basis)

$c_j = \frac{(\chi_{2j-1} + i\chi_{2j})}{\sqrt{2}}$ $S_{\mathrm{EE}}(t) = -\mathrm{Tr}\rho_A(t)\log(\rho_A(t))$



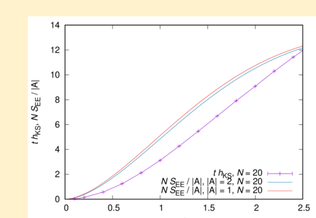
$\rho_A(t) = \mathrm{Tr}_B\rho(t)$, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$



Coarse-grained entropy
= log(# of cells covering the region)
~ (sum of positive $\lambda$) $t$

Kolmogorov-Sinai entropy $h_{\mathrm{KS}}$
= (sum of positive $\lambda$)
= entropy production rate

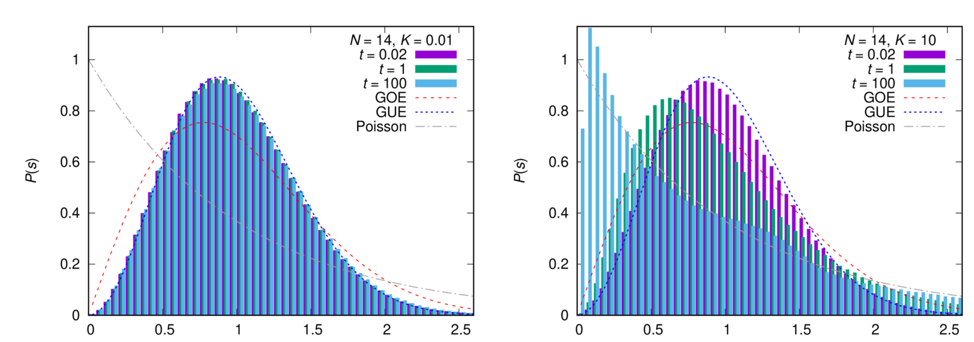
The sum of positive $\lambda_i t$ and entanglement entropy increase with almost the same slope



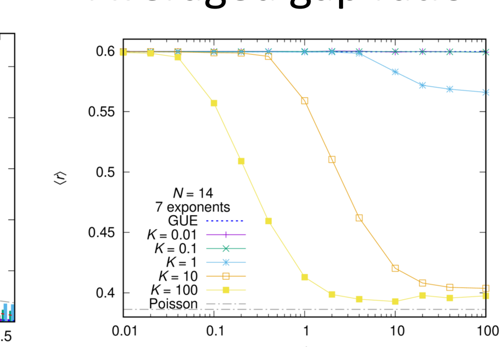
- The difference between $\lambda_N$ and $\lambda_{\mathrm{OTOC}} = \frac{1}{2t}\log\left(\frac{1}{N}\sum_{i=1}^{N} e^{2\lambda_i t}\right) \sim 1/N$
- Also $\lambda_N$ and $\lambda_1$ approach each other ➔ single peak of $\lambda$
- Chaos bound value expected for strong coupling
- Entanglement entropy generation rate $h_{\mathrm{KS}}$ = sum of positive (all) $\lambda_i$

➔ We conjecture that black holes are not only the fastest scramblers [Sekino and Susskind 2008], but also the fastest entropy generators.

(Fixed-*i*) unfolded level separation



Averaged gap ratio



### XXZ spin chain + random field

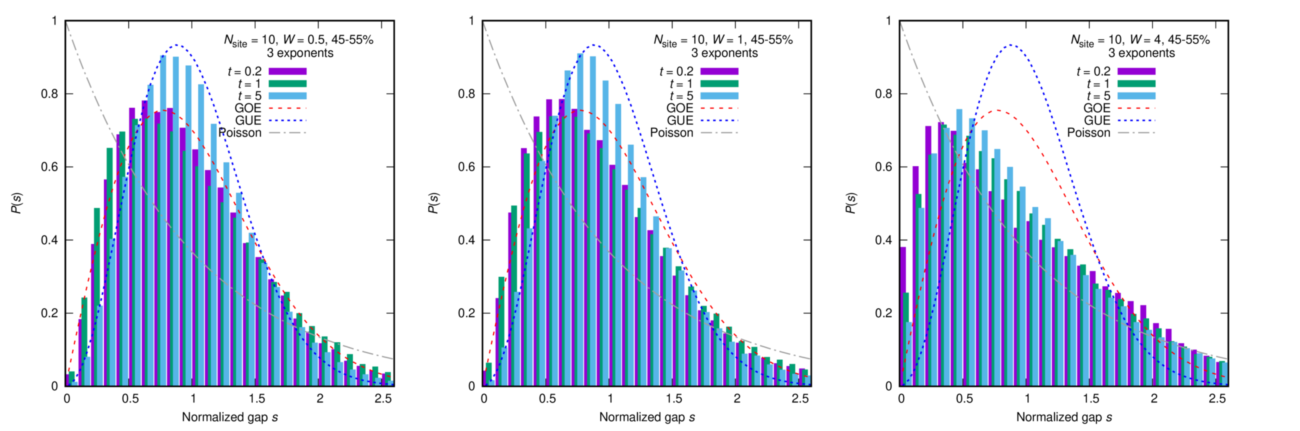
$$\hat{H} = \sum_{i}^{N} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_{i+1} + \sum_{i}^{N} h_i \hat{S}_i^z$$

$h_i$: uniform in $[-W, W]$

Standard model of many-body localization ($W_c \approx 3.5$)

e.g. [D. J. Luitz, N. Laflorencie, and F. Alet, PRB **91**, 081103 (2015)]

$\hat{M}_{ab}(t) = [\hat{S}_a^+(t), \hat{S}_b^-(0)]$



$W = 0.5$: GUE approached $W = 4$: stays close to Poisson

In both models, quantum Lyapunov spectrum distinguishes chaotic and non-chaotic phases.

Chaotic ➔ Universal random matrix behavior

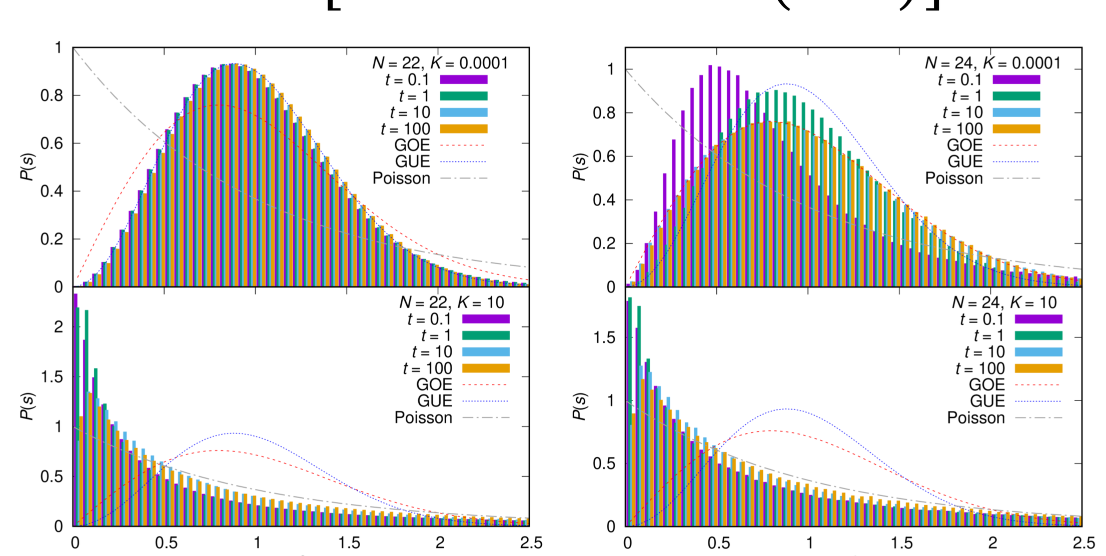
Non-chaotic ➔ Nearly uncorrelated, exp(-*s*) level separation

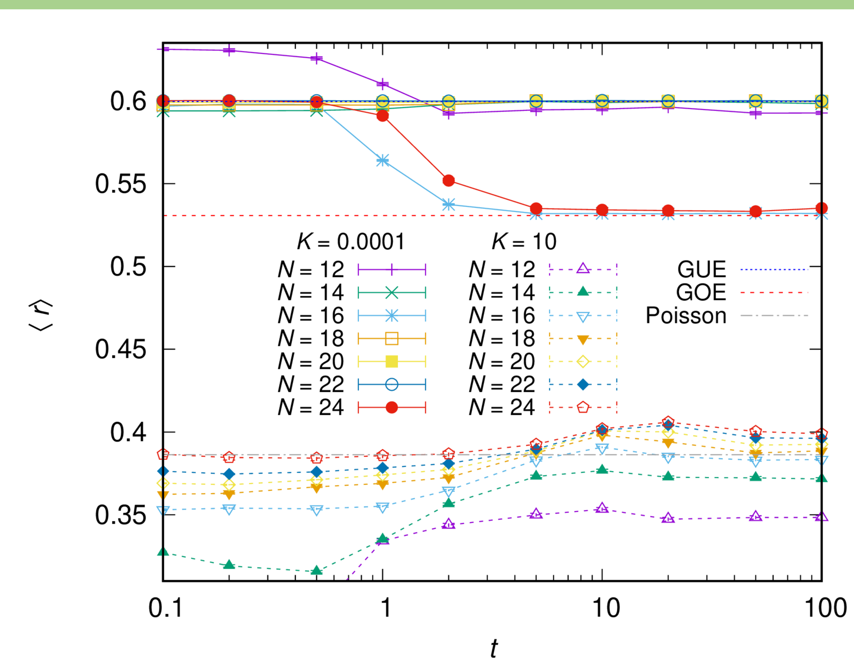
## Two-point correlator matrix (arXiv:1902.11086)

### SYK model

$$G_{ab}^{(\phi)} = \langle \phi | \hat{\chi}_a(t)\hat{\chi}_b(0) | \phi \rangle$$

$$\lambda_k = \log\left[\text{singular values of } \left(G_{ab}^{(\phi)}\right)\right]$$



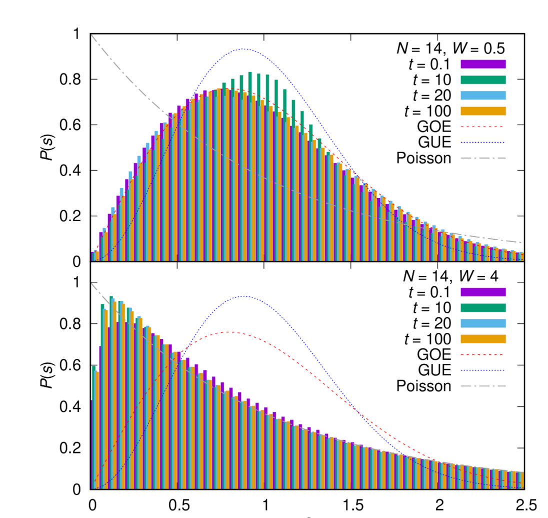


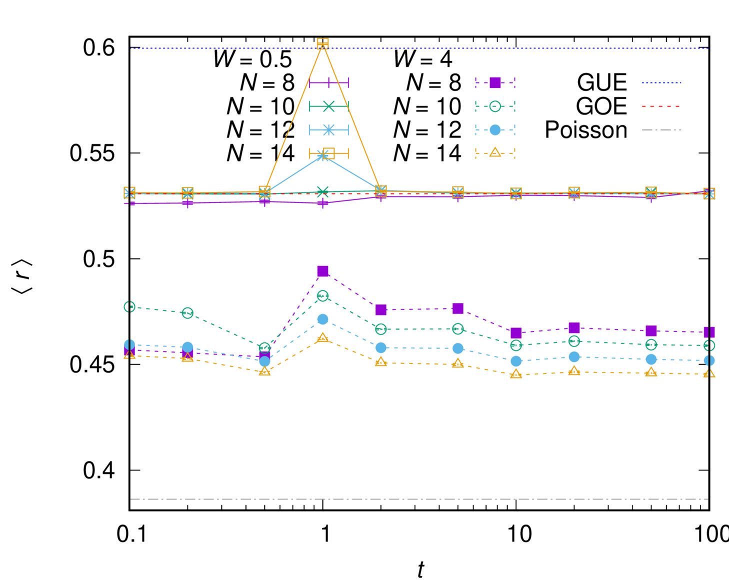
$N$ mod 8 periodicity understood by symmetry of $G_{ab}^{(\phi)}$

(complex symmetric only for $N \equiv 0$ mod 8)

### XXZ spin chain + random field

$$G_{ab}^{(\phi)} = \langle \phi | \hat{\sigma}_a^+(t)\hat{\sigma}_b^-(0) | \phi \rangle$$





Level statistics of $\{\lambda_k\}$: GOE after adequate time

## Summary

We have proposed characterization of quantum chaos by

- **Quantum Lyapunov spectrum**
  - ➢ from analogy to the classical case
  - ➢ Lyapunov growth for finite time for finite N
- **Two-point correlator matrix**
  - ➢ experimentally more accessible

and have demonstrated their random matrix behavior for the SYK model and the XXZ spin chain + random field.